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Strain gradient plasticity, strengthening effects and plastic limit analysis

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ABSTRACT

Within the framework of isotropic strain gradient plasticity, a rate-independent constitutive model exhibiting size dependent hardening is formulated and discussed with particular concern to its strengthening behavior. The latter is modelled as a (fictitious) isotropic hardening featured by a potential which is a *positively degree-one homogeneous function of the effective plastic strain and its gradient*. This potential leads to a *strengthening law* in which the *strengthening stress*, i.e. the increase of the plastically undeformed material initial yield stress, is related to the effective plastic strain through a second order PDE and related higher order boundary conditions. The plasticity flow laws, with the role there played by the strengthening stress, are addressed and shown to admit a maximum dissipation principle. For an idealized elastic perfectly plastic material with strengthening effects, the plastic collapse load problem of a micro/nano scale structure is addressed and its basic features under the light of classical plastic limit analysis are pointed out. It is found that the conceptual framework of classical limit analysis, including the notion of rigid-plastic behavior, remains valid. The lower bound and upper bound theorems of classical limit analysis are extended to strengthening materials. A static-type *maximum principle* and a kinematic-type *minimum principle*, consequences of the lower and upper bound theorems, respectively, are each independently shown to solve the collapse load problem. These principles coincide with their respective classical counterparts in the case of simple material. Comparisons with existing theories are provided. An application of this nonclassical plastic limit analysis to a simple shear model is also presented, in which the plastic collapse load is shown to increase with the decreasing sample size (Hall–Petch size effects).

1. Introduction

As shown in a number of experimental observations, at micro/nano scales materials exhibit a notable size dependence, in the sense that "smaller is stronger". This phenomenon manifests itself in a variety of forms, such as: (i) the increase of the strain hardening rate with the decreasing of the size of metal and alloy specimen, (ii) the increase of the initial yield stress, or strength, with the decreasing of grain size of crystalline materials (the so-called Hall–Petch effect, or "strengthening"), (iii) the increase of strength with the reduction of the reinforcing particle size in composite materials at constant volume fraction (the so-called "dispersion strengthening"), (iv) the increase of the indentation hardness with the decreasing of the indentation depth, etc.; see e.g. Stemalshenko et al. (1993), McElhaney et al. (1998), Fleck et al. (1994), Lloyd (1994), Ma and Clarke (1995), Poole et al. (1996), Stölken and Evans (1998), Huang and Spaepan (2000), Haque and Saif (2003). For an overview of the rich literature on this subject, see Hutchinson (2000), Gudmundson (2004), Hansen (2004), Gurtin and Anand (2005), Abu Al-Rub (2008).

Simulation procedures based on molecular dynamics and discrete dislocation theories have been used to model the above phenomena (see e.g. Van der Giessen and Needleman (1995), Claveringa et al. (1999), Zibib et al. (2002)), but the characteristic structural length in the microsize range is still too large for an effective numerical implementation of these methods (Gurtin and Anand, 2005). The continuum approach via strain gradient plasticity has been more effective and fruitful for its capacity of describing the afore-mentioned size effects. The relevant key concept consists in introducing, into the governing equations, a gradient of a suitable measure of plastic strain, which proves to be a means to carry in the fundamental role played by dislocations, and in particular by Geometrically Necessary Dislocations (GNDs), (Ashby, 1970), within the underlying microstructural deformation processes. The various existing theories of gradient plasticity differ from one another for the ways they incorporate the gradient dependence, but all of them prove to be able to predict and describe, in all or in part, the mentioned size effects.

Aifantis (1984, 1987, 2003) first introduced a strain gradient correction into the standard yield condition in order to address

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strain localization phenomena. A pletora of theories followed the pioneering work by Aifantis, devoted either to crystal models, or to continuum ones. For a revision of these works, see e.g. Fleck and Hutchinson (1997), Hutchinson (2000), Gudmundson (2004), Gurtin and Anand (2005), Kuroda and Tvergaard (2006); see also the special issue of *Modelling and Simulation in Material Science and Engineering, vol. 15(1), 2007*. More recent contributions in the framework of isotropic gradient plasticity have been given by Lele and Anand (2008), Abu Al-Rub (2008), Fleck and Willis (2008), Polizzotto (2007), Polizzotto (2008a), Polizzotto (2009a), Polizzotto (2009b), and in the framework of crystal plasticity by Gurtin et al. (2007), Bardella (2007), Borg (2007).

Among the variety of size effects, strengthening effects have recently been modelled within isotropic gradient plasticity by Fredriksson and Gudmundson (2005) and, independently, by Gurtin and Anand (2005). These authors employ a similar visco-plastic model with a power law of degree $m > 0$ which, as m tends to the pertinent extreme value, simulates a rate-independent material model and thus reproduces strengthening effects exempt from the influence of plastic strain rate. This method was improved for one-dimensional structures (Anand et al., 2005) and for pluri-dimensional ones obeying simplified constitutive laws (Lele and Anand, 2008). Borg (2007) extended the work by Gurtin and Anand (2005) to crystal plasticity; Abu Al-Rub (2008) addressed the influence of a boundary layer upon strengthening effects.

The purpose of the present paper is the formulation of a rate-independent strain gradient plasticity model different from the visco-plastic one previously mentioned, but equally capable to capture strengthening effects. The classical internal variable framework is employed. The key idea consists in introducing a *fictitious isotropic strain hardening* governed by a potential (*strengthening potential*), say ψ_{st}, which is a *positively degree–one homogeneous function of the effective plastic strain and the effective plastic strain gradient* (four strain variables in total), hence the (as many) related thermodynamic forces are functions of the same type, but of *degree zero*. This means that, for a material model with kinematic and isotropic hardening and with strengthening, besides the standard hardening potential(s), one has to further introduce a strengthening potential, as it will be actually done in the present paper. The strengthening potential is sharply different from potentials used to model other types of size effects, like the so-called *energetic size effects* (Gurtin and Anand, 2005), which are in general homogeneous functions analogous to the strengthening potential, but of degree *larger than one* (usually of degree two or more). As it will be shown also by means of a simple numerical example, the proposed model is able to capture and describe strengthening effects in a manner quite similar to the visco-plastic model previously mentioned.

The introduction of a strengthening potential as mentioned above implies that the related thermodynamic forces, here called *primitive strengths* (a scalar and a vector), obey some specific plasticity-like laws. A *strengthening surface* bounds the primitive strengths and an (associative) normality rule provides the work-conjugate four-component generalized effective plastic strain. The primitive strengths, with values on the strengthening surface, constitute the basis for the strengthening law of the material, which is obeyed by the *strengthening stress*, a scalar measuring the increase (or decrease) of the initial strength exhibited by the material during plastic deformation.

A main concern of the present work is the idealized case of a perfectly plastic material that exhibits strengthening effects. A body of this type, if subjected to a monotonically increasing load, is expected to behave like one without strengthening effects in the framework of classical plastic limit analysis. At a certain value of the load, an instantaneous collapse mechanism occurs, while the structure continues to deform plastically at a constant load, the value of which is related to an increased yield strength of the material. Since the amount of this strenthening is not known in advance, as it depends on the shape of the collapse mechanism, the problem of evaluating the collapse load, as well as the related collapse mechanism, is not a trivial one. For one-dimensional structures, problems of this type were addressed by Anand et al. (2005).

The equation set governing the collapse load problem for a given micro/nano scale structure will be addressed and its analogies and differences with respect to the classical limit analysis problem will be pointed out. This problem differs from its classical counterpart due to the presence, in the yield function, of a strengthening stress, which is related to plastic deformation in a complex way (it is a second order differential form in the effective plastic strain and is accompanied by some higher order boundary conditions). Nevertheless, the lower bound and upper bound theorems of classical limit analysis will be extended to materials with strengthening effects, although at the cost of inevitable mathematical complications. The plastic collapse load problem can be solved by addressing the entire set of governing equations, or also by means of one of two variational principles: one is a static–type *maximum principle*, the other is a kinematic–type *minimum principle*. These principles, direct consequences of the lower and upper bound theorems, respectively, are independently shown to solve the collapse load problem.

It is worth noting that the kind of nonclassical plastic limit analysis addressed in this paper is different in nature from other ones existing in the literature. In fact, in the latter kind of limit analyses, Cosserat material models (Mühlhaus, 1989), or polar-type Toupin–Mindlin material models (Zhao et al., 2007), suitably extended to plasticity, are considered. The gradient dependence is there carried into the constitutive model by some higher order plastic strain tensors, but the initial yield stress is size independent, hence no strengthening effects are there involved.

The so-called residual-based strain gradient plasticity theory will be used (Polizzotto and Borino, 1998; Polizzotto, 2003, 2007, 2008a, 2009a,b; Borino and Polizzotto, 2007). For self-containment reasons, we briefly resume its essentials. This theory is centered upon the concept of *(nonlocality energy) residual*, say P. This represents the long distance energy density transmitted to the generic particle of the body from all other particles of it as a consequence of the underlying deformation process, and is identically vanishing only in the case of a simple material. The theory is grounded upon three constitutive assumptions:

(i) The *insulation condition*, which states that no long distance interactions occurs between the body and the exterior ambient, and thus

$$\int_V P\,\mathrm{d}v = 0 \tag{1}$$

for whatever deformation mechanism. If there exist boundary and/or interface layers with surface energy, contributions from these layers should be added to (1) (Polizzotto, 2009b), but this case is excluded here.

(ii) The *bilinear dissipation condition*, consequence of the Onsager reciprocity principle, assumed to hold in the present context, which implies that the dissipation power density, D, takes on a bilinear form in terms of independent driving plastic strain rates, say $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$, and related thermodynamic forces, or affinities, say $\boldsymbol{\rho}$, and thus

$$D = \boldsymbol{\rho} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}. \tag{2}$$

(iii) The *locality recovery condition*, which states that the residual P must vanish in the volume V of the body for whatever uniform-plastic-strain deformation mechanism, such that the material behaves as a simple one correspondingly.

The proposed gradient plasticity model proves to be an extension of the mentioned residual-based one, made capable to capture strengthening effects (Hall–Petch effects) through the introduction of an ad-hoc strengthening potential. It is purely phenomenological in nature and consistent with thermodynamics principles; its connection to real material behavior is ascertained, but only in the extent considered sufficient for the present theoretical research work.

The outline of the paper is as follows. In Section 2, besides the standard hardening potentials, the strengthening potential is introduced and discussed. Also, thermodynamic arguments are used to derive the pertinent restrictions on the constitutive equations together with the expressions of the dissipation density and the energy residual. In Section 3, the plasticity evolution laws in the presence of strengthening effects are presented, showing that a maximum dissipation principle holds. In Section 4 the perfectly plastic material with strengthening effects is addressed and the relevant plastic collapse load problem is discussed under the light of classical limit analysis. In Section 5 the extended lower bound and upper bound theorems are proved, whereas in Section 6 the consequent maximum and minimum principles are independently proved. Section 7 is devoted to comparisons with other theories. An illustrative example of nonclassical limit analysis is presented in Section 8. Conclusions are in Section 9.

Notation. A compact notation is used, with boldface letters denoting vectors or tensors of any order. The scalar product between vectors or tensors is denoted with as many dots as the number of contracted index pairs. For instance, denoting by $\boldsymbol{u} = \{u_i\},\ \boldsymbol{v} = \{v_i\},\ \boldsymbol{\varepsilon} = \{\varepsilon_{ij}\},\ \boldsymbol{\sigma} = \{\sigma_{ij}\},\ \boldsymbol{\tau} = \{\tau_{ijk}\}$ and $\boldsymbol{A} = \{A_{ijkh}\}$ some vectors and tensors, one can write: $\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i,\ \boldsymbol{\sigma}:\boldsymbol{\varepsilon} = \sigma_{ij}\varepsilon_{ji},\ \boldsymbol{A}:\boldsymbol{\varepsilon} = \{A_{ijkh}\varepsilon_{hk}\},\ \boldsymbol{A}\vdots\boldsymbol{\tau} = \{A_{ijkh}\tau_{hkj}\},\ \boldsymbol{A}^T\vdots\boldsymbol{\tau} = \{A_{ijkh}\tau_{ijk}\}$, where $\boldsymbol{A}^T$ is the transpose of $\boldsymbol{A}$. The summation rule for repeated indices holds and the subscripts denote components with respect to an orthogonal Cartesian co-ordinate system, say $\boldsymbol{x} = (x_1, x_2, x_3)$. An upper dot over a symbol denotes its time derivative, $\dot{\boldsymbol{u}} = \partial\boldsymbol{u}/\partial t$. The symbol ∇ denotes the spatial gradient operator, i.e. $\nabla\boldsymbol{u} = \{\partial_i u_j\}$. The symbol $\boldsymbol{e} = \{e_{ijk}\}$ indicates the alternating third order tensor. The symbol := means equality by definition. Other symbols will be defined in the text at their first appearence.

2. Thermodynamic approach

2.1. Premises

Let us consider a (micro/nano scale) solid body of volume V and boundary surface $S := \partial V$, which in its undeformed state is referred to a Cartesian orthogonal co-ordinate system, say $\boldsymbol{x} = (x_1, x_2, x_3)$. The body is subjected to quasi-statically variable loads and experiences small deformations. For simplicity, isothermal conditions are assumed. The constitutive behavior of the material is governed by a Helmholtz free energy potential ψ of the form:

$$\psi = \psi_e(\boldsymbol{\varepsilon}^e) + \psi_{p1}(\boldsymbol{\varepsilon}^p, \nabla\boldsymbol{\varepsilon}^p, \nabla\boldsymbol{w}^p) + \psi_{p2}(r, \nabla r) + \psi_{st}(\kappa, \nabla\kappa) \tag{3}$$

where $\boldsymbol{\varepsilon}^e,\ \boldsymbol{\varepsilon}^p$ denote elastic and plastic parts of the (standard) total strain tensor, $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$; $\boldsymbol{w}^p$ is the skew part of the plastic distortion tensor, $\boldsymbol{H}^p = \boldsymbol{\varepsilon}^p + \boldsymbol{w}^p$, plastic part of the displacement gradient $\boldsymbol{H} := \nabla\boldsymbol{u}$; r and κ denote some scalar measures of plastic strain. The fields $\boldsymbol{\varepsilon}^p,\ r,\ \kappa$ are regular in the whole V. The variables on which $\psi_{p1},\ \psi_{p2}$ and ψ_{st} depend play the role of *internal variables.*

More precisely:

- $\psi_e \ \rightarrow$ *Elastic strain energy*, usually $\psi_e = \frac{1}{2}\boldsymbol{\varepsilon}^e : \boldsymbol{M} : \boldsymbol{\varepsilon}^e$, with $\boldsymbol{M}$ being the elasticity fourth order moduli tensor with its standard symmetry features.
- $\psi_{p1} \ \rightarrow$ *Kinematic hardening potential*, which is a homogeneous function of its arguments, of degree $m \geqslant 2$. Often in the literature ψ_{p1} is considered a function of the GND (geometrically necessary dislocation) density (Ashby, 1970) through the tensor $\nabla\times\boldsymbol{H}^p$ (see e.g. Gurtin (2004)), or of $\nabla\times\boldsymbol{\varepsilon}^p$ (see e.g. Gurtin and Anand (2005), Anand et al. (2005), Lele and Anand (2008)), but here we prefer to adopt the more general functional dependence on $\nabla\boldsymbol{\varepsilon}^p$ and $\nabla\boldsymbol{w}^p$. (*Note:* $\nabla\times\boldsymbol{\varepsilon}^p = \mathrm{curl}\,\boldsymbol{\varepsilon}^p = -\boldsymbol{e}:\nabla\boldsymbol{\varepsilon}^p$, hence $\partial\psi_{p1}/\partial(\nabla\boldsymbol{\varepsilon}^p) = \boldsymbol{e}\cdot(\partial\psi_{p1}/\partial(\nabla\times\boldsymbol{\varepsilon}^p))$, and analogously for $\nabla\boldsymbol{w}^p$.)
- $\psi_{p2} \ \rightarrow$ *Isotropic hardening potential*, which is a homogeneous function of r and ∇r, of degree $m \geqslant 2$. The state variable r is a degree-one positively homogeneous function of the plastic strain, but for the moment the exact relation between r and $\boldsymbol{\varepsilon}^p$ is left unspecified (it will be as soon as the plasticity evolution law is chosen).
- $\psi_{st} \ \rightarrow$ *Strengthening potential*, which is a positively degree-one homogeneous (nonlinear) function of κ and $\nabla\kappa$, where κ is similar to, or even coincident with, r. Since, on removing the dependence of ψ_{st} on $\nabla\kappa$, no strengthening effects have to arise, correspondingly ψ_{st} has to vanish identically for whatever κ. Hence, ψ_{st} has to satisfy the condition:

$$\psi_{st} \rightarrow 0 \quad \text{for } \nabla\kappa \rightarrow \boldsymbol{0}. \tag{4}$$

Possible forms for the above potentials are the following:

$$\psi_{p1} = \frac{1}{2}h_1\left(\|\boldsymbol{\varepsilon}^p\|^2 + \ell_{1(1)}^2\|\nabla\boldsymbol{\varepsilon}^p\|^2 + \ell_{1(2)}^2\|\nabla\boldsymbol{w}^p\|^2\right), \tag{5}$$

$$\psi_{p2} = \frac{1}{2}h_2(r^2 + \ell_2^2\|\nabla r\|^2), \tag{6}$$

$$\psi_{st} = \sigma_0(p - \kappa), \quad p := \sqrt{\kappa^2 + \ell^2\|\nabla\kappa\|^2}, \tag{7}$$

where h_1 and h_2 are hardening (or softening) moduli; $\ell_{1(1)},\ \ell_{1(2)},\ \ell_2$ and ℓ denote internal lengths, and σ_0 is the initial yield stress of the plastically undeformed material. Please note that (7) meets condition (4). Also note that the internal variables r and κ may in practice be taken coincident with each other, but they are considered as distinct variables for more clarity. Although the major interest here is in strengthening effects and the related nonclassical limit analysis, consideration of the standard hardening material behavior is believed necessary for a useful and complete discussion.

2.2. Digression on the strengthening potential

Before going on with the issues of this section, it is convenient to point out some fundamental features of the strengthening potential previously introduced. This potential can be geometrically represented in the four-dimension space $(\kappa,\ \nabla\kappa)$ as a conical surface with its vertex on the strain origin, the shape of which changes with the internal length scale parameter, say ℓ. Like for a dissipation function of flow plasticity (Martin, 1975), the potential ψ_{st} admits meaningful partial derivatives, say

$$Y_0 := \frac{\partial\psi_{st}}{\partial\kappa}, \quad \boldsymbol{Y}_1 := \frac{\partial\psi_{st}}{\partial(\nabla\kappa)}, \tag{8}$$

except at the origin, whereas the quantities $Y_0,\ \boldsymbol{Y}_1$, here called *primitive strengths*, stay in the inside of a (closed) *strengthening surface*, say

$$\phi = \phi(Y_0, \boldsymbol{Y}_1) < 0. \tag{9}$$

A correspondence is established between the primitive strength points, $(Y_0,\ \boldsymbol{Y}_1)$, and the generalized effective plastic strain points, $(\kappa,\ \nabla\kappa)$, which is similar to a deformation-theory plasticity law. Namely, there exists a scalar $\mu = \mu(\kappa,\ \nabla\kappa)$ such that:

$$\left.\begin{array}{ll} \kappa = \mu\partial\phi/\partial Y_0, & \nabla\kappa = \mu\partial\phi/\partial\boldsymbol{Y}_1 \\ \phi(Y_0, \boldsymbol{Y}_1) \leqslant 0, & \mu \geqslant 0, \quad \mu\phi = 0 \end{array}\right\} \quad \text{in } V. \tag{10}$$

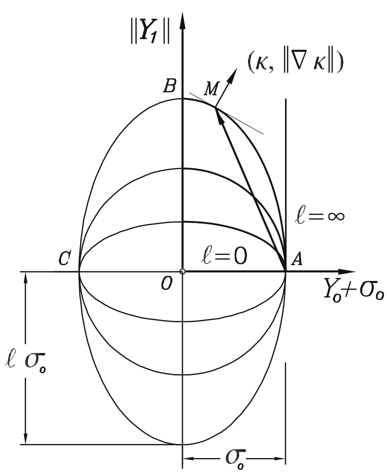

Fig. 1. Family of strengthening ellipses generated by varying the internal length scale parameter, ℓ.

In the case of a strengthening potential as in (7), the following can be found:

$$\left.\begin{array}{l} Y_0 = \sigma_0\left(\frac{\kappa}{p} - 1\right), \quad \boldsymbol{Y}_1 = \ell^2\sigma_0\frac{\nabla\kappa}{p} \\ \phi = \sqrt{(Y_0+\sigma_0)^2 + \frac{1}{\ell^2}\|\boldsymbol{Y}_1\|^2} - \sigma_0 \leqslant 0 \\ \mu = p \quad \text{for } \phi = 0, \quad \mu = 0 \quad \text{for } \phi < 0 \end{array}\right\} \quad \text{in } V, \tag{11}$$

where p is given in (7).

The strengthening surface $\phi = 0$ can be represented through its projection on the plane with co-ordinates $Y_0 + \sigma_0,\ \|\boldsymbol{Y}_1\|$. It has the shape of an ellipse with semidiameters σ_0 and $\ell\sigma_0$, Fig. 1. Primitive strengths values in the interior of the strengthening surface ($\phi < 0$) pertain to the material being plastically undeformed. Primitive strength values such that $\phi = 0$ instead characterize a strengthened state of the material, which is featured by the generalized effective plastic strains $(\kappa,\ \nabla\kappa)$ displayed by the outward normal to the strengthening surface at the relevant primitive strength point.

A family of ellipses is generated with varying ℓ, with diameters OA fixed, but OB increasing with ℓ. The amplitude of the primitive strength, AM, is allowed to take on higher values with higher ℓ values, or, at a constant ℓ, with larger values of $\|\nabla\kappa\|$. As it will be clearer shortly, the primitive strengths play a crucial role in the strengthening process occurring during plastic deformation.

An interpretation of the primitive strengths from the microstructural point of view can be attempted. Namely, κ and $\nabla\kappa$ can be thought of to represent, respectively, the SSD (statistically stored dislocation) density and the GND (geometrically necessary dislocation) density (Ashby, 1970), whereas the primitive strengths, Y_0 and $\boldsymbol{Y}_1$, can be thought of to represent the respective associated Peach–Koehler forces. As long as the latter forces are below a certain threshold (here interpreted by the strengthening surface), hence the material is still plastically undeformed, no strengthening effects arise. Instead, if the primitive strengths are at the strengthening limit, hence the material is already plastically deformed, then strengthening effects do manifest themselves. The latter effects are macroscopically described by a *strengthening law* of the material which will be discussed shortly.

2.3. Restrictions on the constitutive equations

The thermodynamic restrictions on the constitutive equations are derived in this subsection. As usual with the residual-based gradient plasticity theory, the starting point is the (nonlocal) Clausius–Duhem inequality, which (in the assumed isothermal conditions) reads:

$$D := \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\psi} + P \geqslant 0 \quad \text{in } V, \tag{12}$$

where D denotes the dissipation power density. Let us introduce the notation:

$$\boldsymbol{X}_0 := \frac{\partial\psi_{\mathrm{p1}}}{\partial\boldsymbol{\varepsilon}^{\mathrm{p}}}, \quad \boldsymbol{X}_1' := \frac{\partial\psi_{\mathrm{p1}}}{\partial(\nabla\boldsymbol{\varepsilon}^{\mathrm{p}})}, \quad \boldsymbol{X}_1'' := \frac{\partial\psi_{\mathrm{p1}}}{\partial(\nabla\boldsymbol{w}^{\mathrm{p}})} \tag{13}$$

$$R_0 := \frac{\partial\psi_{\mathrm{p2}}}{\partial r}, \quad \boldsymbol{R}_1 := \frac{\partial\psi_{\mathrm{p2}}}{\partial(\nabla r)}, \tag{14}$$

where the third order tensors

$$\boldsymbol{X}_1' = \left\{X_{1kij}'\right\} = \left\{\frac{\partial\psi_{\mathrm{p1}}}{\partial(\partial_k\varepsilon_{ij}^{\mathrm{p}})}\right\}, \quad \boldsymbol{X}_1'' = \left\{X_{1kij}''\right\} = \left\{\frac{\partial\psi_{\mathrm{p1}}}{\partial(\partial_k w_{ij}^{\mathrm{p}})}\right\} \tag{15}$$

are, respectively, symmetric and antisymmetric with respect to the index pair (i,j). Also, let us assume a separate dependency of ψ_{p1} on $\boldsymbol{w}^{\mathrm{p}}$ and let us further pose:

$$\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{\varepsilon}^{\mathrm{p}}, \nabla\boldsymbol{\varepsilon}^{\mathrm{p}}) := \boldsymbol{X}_0 - \nabla\cdot\boldsymbol{X}_1' \tag{16}$$

$$\boldsymbol{\Pi} = \boldsymbol{\Pi}(\nabla\boldsymbol{w}^{\mathrm{p}}) := -\nabla\cdot\boldsymbol{X}_1'' \tag{17}$$

$$R = R(r, \nabla r) := R_0 - \nabla\cdot\boldsymbol{R}_1 \tag{18}$$

$$Y = Y(\kappa, \nabla\kappa) := Y_0 - \nabla\cdot\boldsymbol{Y}_1, \tag{19}$$

where Y_0 and $\boldsymbol{Y}_1$ are the primitive strengths previously introduced. With the above notation in mind and expanding the time derivative of ψ, inequality (12) can be rewritten as:

$$\begin{aligned} D = {} & \left(\boldsymbol{\sigma} - \frac{\partial\psi_{\mathrm{e}}}{\partial\boldsymbol{\varepsilon}^{\mathrm{e}}}\right) : \dot{\boldsymbol{\varepsilon}}^{\mathrm{e}} + (\boldsymbol{\sigma} - \boldsymbol{X}) : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} + \boldsymbol{\Pi} : \dot{\boldsymbol{w}}^{\mathrm{p}} - R\dot{r} - Y\dot{\kappa} \\ & - \nabla\cdot\left(\boldsymbol{X}_1' : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} + \boldsymbol{X}_1'' : \dot{\boldsymbol{w}}^{\mathrm{p}}\right) - \nabla\cdot(\boldsymbol{R}_1\dot{r}) \\ & - \nabla\cdot(\boldsymbol{Y}_1\dot{\kappa}) + P \geqslant 0 \quad \text{in } V. \end{aligned} \tag{20}$$

Since inequality (20) must be satisfied for whatever elastic–plastic deformation mechanism, including purely elastic ones (for which $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \dot{\boldsymbol{w}}^{\mathrm{p}} = \boldsymbol{0},\ \dot{\kappa} = 0,\ P = 0$), (20) implies the equality:

$$\boldsymbol{\sigma} = \frac{\partial\psi_{\mathrm{e}}}{\partial\boldsymbol{\varepsilon}^{\mathrm{e}}} \quad \text{in } V \tag{21}$$

which is the elasticity stress–strain relation. Under the assumption that (21) holds also for general elastic–plastic deformation mechanisms, (20) simplifies as follows:

$$\begin{aligned} D = {} & (\boldsymbol{\sigma} - \boldsymbol{X}) : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} + \boldsymbol{\Pi} : \dot{\boldsymbol{w}}^{\mathrm{p}} - R\dot{r} - Y\dot{\kappa} - \nabla\cdot\left(\boldsymbol{X}_1' : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} + \boldsymbol{X}_1'' : \dot{\boldsymbol{w}}^{\mathrm{p}}\right) \\ & - \nabla\cdot(\boldsymbol{R}_1\dot{r}) - \nabla\cdot(\boldsymbol{Y}_1\dot{\kappa}) + P \geqslant 0 \quad \text{in } V. \end{aligned} \tag{22}$$

By the *bilinear dissipation condition* (2), the plastic power density is here expected to take on a form as

$$D = \hat{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} + \widehat{\boldsymbol{\Pi}} : \dot{\boldsymbol{w}}^{\mathrm{p}} - \widehat{R}\dot{r} - \widehat{Y}\dot{\kappa}, \tag{23}$$

where $\hat{\boldsymbol{\sigma}},\ \widehat{\boldsymbol{\Pi}},\ \widehat{R}$ and $\widehat{Y}$ denote thermodynamic forces work-conjugate of the independent plastic strain rates, i.e. $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}},\ \dot{\boldsymbol{w}}^{\mathrm{p}},\ \dot{r}$ and $\dot{\kappa}$ (11 independent plastic strain rate components in total). As a consequence, substituting from (23) into (22) gives

$$\begin{aligned} P = {} & (\hat{\boldsymbol{\sigma}} - \boldsymbol{\sigma} + \boldsymbol{X}) : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} + (\widehat{\boldsymbol{\Pi}} - \boldsymbol{\Pi}) : \dot{\boldsymbol{w}}^{\mathrm{p}} - (\widehat{R} - R)\dot{r} - (\widehat{Y} - Y)\dot{\kappa} \\ & + \nabla\cdot\left(\boldsymbol{X}_1' : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} + \boldsymbol{X}_1'' : \dot{\boldsymbol{w}}^{\mathrm{p}}\right) + \nabla\cdot(\boldsymbol{R}_1\dot{r}) + \nabla\cdot(\boldsymbol{Y}_1\dot{\kappa}) \quad \text{in } V. \end{aligned} \tag{24}$$

Next, by the *insulation condition* (in the assumed hypothesis that no surfacial energy effects exist), we can write, after application of the divergence theorem:

$$\begin{aligned} & \int_V (\hat{\boldsymbol{\sigma}} - \boldsymbol{\sigma} + \boldsymbol{X}) : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}\,\mathrm{d}v + \int_V (\widehat{\boldsymbol{\Pi}} - \boldsymbol{\Pi}) : \dot{\boldsymbol{w}}^{\mathrm{p}}\,\mathrm{d}v \\ & - \int_V [(\widehat{R} - R)\dot{r} + (\widehat{Y} - Y)\dot{\kappa}]\,\mathrm{d}v \\ & + \int_S \boldsymbol{n}\cdot\left[\boldsymbol{X}_1' : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} + \boldsymbol{X}_1'' : \dot{\boldsymbol{w}}^{\mathrm{p}} + \boldsymbol{R}_1\dot{r} + \boldsymbol{Y}_1\dot{\kappa}\right]\mathrm{d}a = 0, \end{aligned} \tag{25}$$

where $\boldsymbol{n}$ denotes the outward unit vector normal to S. This equality is an identity to be satisfied for whatever elastic–plastic deformation mechanism and evolution law, hence for arbitrary choices of the fields $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$, $\dot{\boldsymbol{w}}^{\mathrm{p}}$ and $\dot{\kappa}$, it thus encompasses the following field and boundary equations:

$$\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \boldsymbol{X}, \quad \widehat{\boldsymbol{\Pi}} = \boldsymbol{\Pi}, \quad \widehat{R} = R, \quad \widehat{Y} = Y \quad \text{in } V \tag{26}$$

$$\boldsymbol{n} \cdot \left(\boldsymbol{X}_1' : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} + \boldsymbol{X}_1'' : \dot{\boldsymbol{w}}^{\mathrm{p}}\right) + \boldsymbol{n} \cdot (\boldsymbol{R}_1 \dot{r} + \boldsymbol{Y}_1 \dot{\kappa}) = 0 \quad \text{on } S. \tag{27}$$

Therefore, by (23) and (24), the expressions of the dissipation D and of the residual P take on the forms

$$D = (\boldsymbol{\sigma} - \boldsymbol{X}) : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} + \boldsymbol{\Pi} : \dot{\boldsymbol{w}}^{\mathrm{p}} - R\dot{r} - Y\dot{\kappa} \geqslant 0 \quad \text{in } V \tag{28}$$

$$P = \nabla \cdot \left(\boldsymbol{X}_1' : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} + \boldsymbol{X}_1'' : \dot{\boldsymbol{w}}^{\mathrm{p}}\right) + \nabla \cdot (\boldsymbol{R}_1 \dot{r}) + \nabla \cdot (\boldsymbol{Y}_1 \dot{\kappa}) \quad \text{in } V. \tag{29}$$

It thus results that the constitutive model exhibits a *kinematic hardening* behavior, which is described by the (total) back-stress tensors $\boldsymbol{X}$ (symmetric) and $\boldsymbol{\Pi}$ (antisymmetric); it also exhibits two types of *isotropic hardening*, which are driven, respectively, by the effective plastic strain variables, r and κ, and described by the (total) drug stresses, R and Y. The latter scalar variables in general obey sharply different state equations, hence they may differ considerably from each other, even in the case that $r \equiv \kappa$. As a consequence of our previous assumptions, both variables are homogeneous functions, R of degree $m-1 \geqslant 1$, Y of degree zero. It thus follows that, if $\kappa = \kappa(\boldsymbol{x})$ and $r = r(\boldsymbol{x})$ are generic fields of effective plastic strains in V, on multiplying both of them by an arbitrary constant scalar, say $\alpha > 0$, one can write:

$$\left.\begin{array}{l} R(\alpha r, \alpha \nabla r) = \alpha^{m-1} R(r, \nabla r) \\ Y(\alpha \kappa, \alpha \nabla \kappa) = Y(\kappa, \nabla \kappa) \end{array}\right\} \quad \text{in } V \quad \text{and} \quad \forall \alpha > 0. \tag{30}$$

The above means that, at difference of R, Y is insensitive to the amplitude of the effective plastic strain and can thus be viewed as an *addition to the initial yield strength* of the material. (This remains true also if the $\boldsymbol{\varepsilon}^{\mathrm{p}}$ field is multiplied by an arbitrary α, since then it is $\kappa(\alpha\boldsymbol{\varepsilon}^{\mathrm{p}}) = |\alpha|\kappa(\varepsilon^{\mathrm{p}})$ and analogously for r.) The isotropic hardening associated to Y can therefore be considered as a (fictitious) one capable to simulate strengthening effects. For this reason ψ_{st} is named *strengthening potential* and Y *strengthening stress (or force)*.

Eq. (27) provides the higher order boundary conditions. Since the higher order stresses $\boldsymbol{X}_1'$, $\boldsymbol{X}_1''$, $\boldsymbol{R}_1$, $\boldsymbol{Y}_1$ relate to hardening constitutive behaviors modelled independently of one another, the addends of (27) must vanish all together, so obtaining the pertinent higher order boundary conditions. These read:

$$\left.\begin{array}{l} \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \dot{\boldsymbol{w}}^{\mathrm{p}} = \boldsymbol{0}, \quad \dot{r} = \dot{\kappa} = 0 \quad \text{on } S_{\mathrm{c}}^{(1)} \\ \boldsymbol{n} \cdot \boldsymbol{X}_1' = \boldsymbol{n} \cdot \boldsymbol{X}_1'' = \boldsymbol{0}, \quad \boldsymbol{n} \cdot \boldsymbol{R}_1 = \boldsymbol{n} \cdot \boldsymbol{Y}_1 = 0 \quad \text{on } S_{\mathrm{f}}^{(1)} \end{array}\right\} \tag{31}$$

Here, $S_{\mathrm{c}}^{(1)}$ denotes a part of S where some nonstandard constraints are located, which impede the onset of plastic strains (with consequent dislocation pile-ups); whereas $S_{\mathrm{f}}^{(1)}$ denotes the complementary part over S, i.e. $S_{\mathrm{f}}^{(1)} := S \setminus S_{\mathrm{c}}^{(1)}$, where no such constraint are located (hence dislocations can move freely outside and plastic strains are unconstrained). Please note that the boundary conditions (31) hold both in rate and time-finite forms (they are related to the fixed surface S). Also note that the above surface decomposition, $S = S_{\mathrm{c}}^{(1)} \cup S_{\mathrm{f}}^{(1)}$, is not necessarily coincident with the standard decomposition $S = S_{\mathrm{c}} \cup S_{\mathrm{f}}$, where S_{c} is the portion of S where the displacements are assigned, S_{f} the one where the tractions are assigned. The boundary conditions (31) are typical for *hard* and *free* boundary surfaces (Gurtin, 2004; Gurtin and Anand, 2005; Polizzotto, 2007). Intermediate forms of higher order boundary conditions for either *stiff* or *soft* boundary surfaces (Fredriksson and Gudmundson, 2005; Abu Al-Rub, 2008; Polizzotto, 2009b) are absent here due to the assumption of no surfacial energy effects.

The *locality recovery condition*, by (29), leads to the requirement that the higher order stresses $\boldsymbol{X}_1'$, $\boldsymbol{X}_1''$, $\boldsymbol{R}_1$ and the higher order primitive strength $\boldsymbol{Y}_1$ be each identically vanishing for whatever uniform-plastic-strain deformation mechanism. This requirement imposes restrictions upon the free energy potential ψ, which we assume to be satisfied (Polizzotto, 2009c).

At the end of the present section, the constitutive model at hand is featured by the state Eqs. (13)–(19) and (21), the dissipation and residual expressions (28) and (29). Eqs. (16)–(19) are PDE systems each of which can in principle be solved independently to express the $\boldsymbol{\varepsilon}^{\mathrm{p}}$, $\boldsymbol{w}^{\mathrm{p}}$, r and κ fields in terms of the $\boldsymbol{X}$, $\boldsymbol{\Pi}$, R and Y fields, provided the appropriate higher order boundary conditions in (31) are accounted for.

3. Evolution laws

On the basis of (28), in the hypothesis of rate-independent associative plasticity, the evolution laws of the material can be set, using a standard notation, as follows:

$$\left.\begin{array}{l} f := g(\hat{\boldsymbol{\sigma}}, \boldsymbol{\Pi}) - R - Y - \sigma_0 \leqslant 0, \quad \dot{\lambda} \geqslant 0, \quad \dot{\lambda} f = 0 \\ \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \dot{\lambda}\partial g/\partial \hat{\boldsymbol{\sigma}}, \quad \dot{\boldsymbol{w}}^{\mathrm{p}} = \dot{\lambda}\partial g/\partial \boldsymbol{\Pi}, \quad \dot{r} = \dot{\kappa} = \dot{\lambda} \end{array}\right\}, \tag{32}$$

where $\dot{\lambda}$ is the plastic activation coefficient and g is a positive definite degree-one homogeneous function of the (symmetric, relative) stress tensor $\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \boldsymbol{X}$ and the (antisymmetric) stress tensor $\boldsymbol{\Pi}$. The effective plastic strain rates, $\dot{r}$ and $\dot{\kappa}$, coincide with each other and are related to both the plastic strain rate, $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$, and the plastic spin, $\dot{\boldsymbol{w}}^{\mathrm{p}}$, that is

$$\dot{\kappa} = \left(\|\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}\|^2 + \|\dot{\boldsymbol{w}}^{\mathrm{p}}\|^2\right)^{1/2}, \tag{33}$$

where we have assumed that

$$\left\|\frac{\partial g}{\partial \hat{\boldsymbol{\sigma}}}\right\|^2 + \left\|\frac{\partial g}{\partial \boldsymbol{\Pi}}\right\|^2 = 1. \tag{34}$$

The evolution laws (32) admit a *maximum dissipation principle*, which reads:

$$\left.\begin{array}{l} D(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{w}}^{\mathrm{p}}, \dot{\kappa}) = \max\limits_{(\hat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\Pi}}, \widehat{R})} \left(\hat{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} + \widehat{\boldsymbol{\Pi}} : \dot{\boldsymbol{w}}^{\mathrm{p}} - \widehat{R}\dot{\kappa}\right) \\ \quad \text{subject to } g(\hat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\Pi}}) - \widehat{R} - \sigma_0 \leqslant 0 \end{array}\right\}, \tag{35}$$

where we have set $\hat{\boldsymbol{\sigma}} := \boldsymbol{\sigma} - \boldsymbol{X}$, $\widehat{\boldsymbol{\Pi}} := \boldsymbol{\Pi}$ and $\widehat{R} := R + Y$. The latter theorem gives the stress state ($\hat{\boldsymbol{\sigma}}$, $\widehat{\boldsymbol{\Pi}}$, $\widehat{R}$) under which the generic material element is able to experience a specified plastic deformation mechanism, i.e. ($\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$, $\dot{\boldsymbol{w}}^{\mathrm{p}}$, $\dot{\kappa}$). It can be easily shown that the Euler–Lagrange equations of (35) coincide with the evolution laws (32). The optimal objective function of (35), D, has the same features as in classical plasticity, and in particular satisfies the equalities:

$$\boldsymbol{\sigma} - \boldsymbol{X} = \frac{\partial D}{\partial \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}}, \quad \boldsymbol{\Pi} = \frac{\partial D}{\partial \dot{\boldsymbol{w}}^{\mathrm{p}}}, \quad R + Y = -\frac{\partial D}{\partial \dot{\kappa}} \tag{36}$$

which hold if, and only if, the plastic deformation mechanism is a nontrivial one; otherwise, (36) loses meaning, but $g(\boldsymbol{\sigma} - \boldsymbol{X}, \boldsymbol{\Pi}) - R - Y - \sigma_0 < 0$ correspondingly.

Let us note that the nonsimple nature of the material does not emerge through the theorem (35), but it does if, in addition to the stress state, one wishes to evaluate the related strain state. Since the reasoning to be used for the proof of the above is similar to one used in an analogous circumstance (Polizzotto, 2007), this point is skipped here for brevity.

It is worth remarking that the flow rule in (32) does depend on the actual stress and plastic strain states of the material. If, at a given state in the body, the plastic strain field is scaled by some $\alpha > 0$, such that

$$\boldsymbol{\varepsilon}^{\mathrm{p}} \to \alpha\boldsymbol{\varepsilon}^{\mathrm{p}}, \quad \boldsymbol{w}^{\mathrm{p}} \to \alpha\boldsymbol{w}^{\mathrm{p}}, \quad \kappa \to \alpha\kappa \tag{37}$$

and thus

$$\boldsymbol{X} \to \alpha^{m-1}\boldsymbol{X}, \quad \boldsymbol{\Pi} \to \alpha^{m-1}\boldsymbol{\Pi}, \quad R \to \alpha^{m-1}R, \quad Y \to Y, \tag{38}$$

we can write, correspondingly,

$$f \to g(\boldsymbol{\sigma} - \alpha^{m-1}\boldsymbol{X}, \alpha^{m-1}\boldsymbol{\Pi}) - \alpha^{m-1}R - Y - \sigma_0 \leqslant 0 \quad \text{in } V. \tag{39}$$

This means that the sum $\sigma_y := Y + \sigma_0$ can be regarded as the *actual initial yield strength* of the material, which is larger than σ_0 if $Y > 0$ (*strengthening*), but smaller than σ_0 if $Y < 0$ (*weakening*). In the following, the word "strengthening" will be used cumulatively.

It is worthwhile to note that, if ψ_{st} was a function of plastic strain, $\boldsymbol{\varepsilon}^p$, instead of the effective plastic strain, κ, the strengthening effects would be simulated by a (fictitious) kinematic hardening instead of an isotropic one. This would have not only just nominal consequences, since in fact in the assumed hypothesis, Eqs. (32)–(35) would continue to hold, but with Y expressed as

$$Y = \boldsymbol{Z} : \boldsymbol{N}, \tag{40}$$

where $\boldsymbol{N} := \dot{\boldsymbol{\varepsilon}}^p / \|\dot{\boldsymbol{\varepsilon}}^p\|, \ \|\boldsymbol{N}\| = 1$ and

$$\left.\begin{array}{l} \boldsymbol{Z}_0 := \partial\psi_{st}/\partial\boldsymbol{\varepsilon}^p, \quad \boldsymbol{Z}_1 := \partial\psi_{st}/\partial(\nabla\boldsymbol{\varepsilon}^p) \\ \boldsymbol{Z} := \boldsymbol{Z}_0 - \nabla \cdot \boldsymbol{Z}_1 \end{array}\right\}. \tag{41}$$

These relations imply that, on scaling $\boldsymbol{\varepsilon}^p$ and $\dot{\boldsymbol{\varepsilon}}^p$ by an arbitrary factor α, the strengthening force Y changes into $\text{sign}(\alpha)Y$, and this in turn implies that, on reversing the sign of $\dot{\boldsymbol{\varepsilon}}^p$, the strengthening effect changes into weakening and viceversa. This is an unpleasant behavior which is avoided by considering ψ_{st} depending on the effective plastic strain instead of the plastic strain, as we do in the present paper.

In closing this section, let us note that the evolution laws (32), together with the state equations (13)–(19) and (21), besides the equilibrium and compatibility equations, govern the relevant boundary–value problem for a given micro/nano scale structure subjected to given loads and specified initial conditions. (*Note*: The equilibrium and compatibility equations remain exactly the same as in classical solid mechanics.) The constitutive relations (8), holding only if the plastic strains are not identically vanishing, can be effectively replaced by (11). We assume that this problem admits a solution, at least in principle. The solution to the corresponding rate problem can be characterized by a minimum principle, extension to gradient plasticity of an analogous principle of classical flow plasticity theory. Such extension can be achieved following a reasoning path quite similar to one in (Polizzotto, 2007) and it is thus skipped for brevity. An analogous principle for deformation-theory plasticity can be equally proved, following (Fleck and Willis, 2008; Polizzotto, 2007).

4. Perfectly plastic material with strengthening effects

4.1. General

If in the treatment of the preceding sections we suppress ψ_{p1} and ψ_{p2}, no kinematic, nor isotropic hardening effects are exhibited by the material model, except for the fictitious isotropic hardening that simulates strengthening. The resulting material proves to be a perfectly plastic one capable to manifest strengthening effects during deformation. Such a material model is quite interesting because, as observed by Anand et al. (2005), it enables us to address material strengthening effects under the light of classical plastic limit analysis.

Under the above assumption, the developments of Sections 2 and 3 continue to hold, but with $\psi_{p1} = \psi_{p2} = 0$ and thus Eqs. (13)–(18) loose meaning, whereas Eqs. (8), (19), (28), (29), (31) and (32) become, respectively:

$$D = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p - Y\dot{\kappa} \geqslant 0 \quad \text{in } V \tag{42}$$

$$P = \nabla \cdot (\boldsymbol{Y}_1\dot{\kappa}) \quad \text{in } V \tag{43}$$

$$\left.\begin{array}{l} Y = Y(\kappa, \nabla\kappa) := Y_0 - \nabla \cdot \boldsymbol{Y}_1 \quad \text{in } V \\ \dot{\kappa} = 0 \quad \text{on } S_c^{(1)}, \quad \boldsymbol{n} \cdot \boldsymbol{Y}_1 = 0 \quad \text{on } S_f^{(1)} \end{array}\right\} \tag{44}$$

$$\left.\begin{array}{l} f = g(\boldsymbol{\sigma}) - Y - \sigma_0 \leqslant 0, \quad \dot{\lambda} \geqslant 0, \quad \dot{\lambda}f = 0 \\ \dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\partial g/\partial\boldsymbol{\sigma}, \quad \dot{\kappa} = \dot{\lambda} \end{array}\right\} \quad \text{in } V. \tag{45}$$

Here, Eq. $(44)_1$ coincides with (19) and is here reported again for more clarity. Eqs. (33)–(36) are still valid, but with $\boldsymbol{X} = \boldsymbol{\Pi} \equiv \boldsymbol{0}$, $R \equiv 0$, $\dot{\boldsymbol{w}}^p \equiv \boldsymbol{0}$, hence $\hat{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma}$.

In the conditions being considered, the body is expected to exhibit a behavior similar to the one of a classical perfectly plastic material structure under monotonically increasing load. At a certain value of this load, an instantaneous collapse mechanism occurs and the load stops increasing further, while the structure continues to deform plastically at a constant load, the value of which is related to the increased yield strength of the material. Since, as already noted in Section 1, the amount of this strengthening is not known in advance, we have to evaluate it by solving a specific problem of a nonclassical plastic limit analysis for the strengthening material.

In relation to this problem, a few general considerations can be advanced at this point. Let the structure find itself in a limit state of collapse and let $\dot{\boldsymbol{\varepsilon}}^p, \ \dot{\kappa}$ describe the relevant collapse mechanism. At points of $V_p \subseteq V$ where the yield condition is reached, we have by (45):

$$Y = g(\boldsymbol{\sigma}) - \sigma_0 \quad \text{in } V_p \subseteq V, \tag{46}$$

which holds during the unconstrained deformation. Since $\dot{\boldsymbol{\sigma}} = \boldsymbol{0}$ everywhere, it follows that *the strengthening stress has to be time-independent in the limit state*, $\dot{Y} = 0$. Also, let us remember that Y is a positively degree-zero homogeneous function of κ and $\nabla\kappa$ and that κ is expressed by the last equality in $(45)_5$, which is equivalent to

$$\kappa(\boldsymbol{x}, t) = \int_0^t \dot{\lambda}(\boldsymbol{x}, \bar{t})\, \mathrm{d}\bar{t} \quad \forall \boldsymbol{x} \in V, \tag{47}$$

where $\kappa(\boldsymbol{x}, 0) \equiv 0$ by assumption. A necessary and sufficient condition in order that the time variable t disappear as an (implicit) argument of Y is that $\dot{\lambda}$ be time independent, i.e. $\dot{\lambda} \equiv \Lambda(\boldsymbol{x})$, such that (47) becomes

$$\kappa(\boldsymbol{x}, t) = t\Lambda(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in V. \tag{48}$$

Correspondingly, $\psi_{st} = \psi_{st}(\Lambda, \nabla\Lambda)$ and, in the region $V_p \subseteq V$ where $\Lambda \not\equiv 0$, the primitive strengths are expressed as

$$Y_0 = \frac{\partial\psi_{st}}{\partial\Lambda}, \quad \boldsymbol{Y}_1 = \frac{\partial\psi_{st}}{\partial(\nabla\Lambda)}, \quad \text{in } V_p \subseteq V, \tag{49}$$

whereas the strengthening stress, Y, is given by

$$Y = Y(\Lambda, \nabla\Lambda) := Y_0(\Lambda, \nabla\Lambda) - \nabla \cdot \boldsymbol{Y}_1(\Lambda, \nabla\Lambda) \quad \text{in } V_p \subseteq V. \tag{50}$$

The latter equation is a second order PDE in the space variable Λ, to be used to evaluate the shape of the collapse mechanism together with the related strengthening stress distribution. For this purpose, the higher order boundary conditions in (44) must be accounted for, cast in the equivalent form

$$\Lambda = 0 \quad \text{on } S_{pc}^{(1)}, \quad \boldsymbol{n} \cdot \boldsymbol{Y}_1(\Lambda, \nabla\Lambda) = 0 \quad \text{on } S_{pf}^{(1)}, \tag{51}$$

where $S_{pc}^{(1)}$ and $S_{pf}^{(1)}$ denote the portions of $S_c^{(1)}$ $S_f^{(1)}$ lying on $S_p = \partial V_p$. The higher order boundary conditions upon $S_{p(int)}$, the internal surface that separates V_p from the rest of the body (if nonempty) must be inforced as

$$\Lambda = \boldsymbol{n} \cdot \boldsymbol{Y}_1 = 0 \quad \text{on } S_{p(int)}. \tag{52}$$

The plasticity-like laws of (10) remain formally the same, but with κ replaced by Λ. Obviously it is $\phi \equiv 0$ in V_p, but $\phi < 0$ in $V \setminus V_p$.

It is useful to note that the ψ_{st} of (7) here takes on the form

$$\psi_{st} = \sigma_0(p - \Lambda), \quad p := \sqrt{\Lambda^2 + \ell^2\|\nabla\Lambda\|^2}, \tag{53}$$

that correspondingly the primitive strengths and the strengthening stress are expressed as

$$Y_0 = \sigma_0\left(\frac{\Lambda}{p} - 1\right), \quad \boldsymbol{Y}_1 = \ell^2\sigma_0\frac{\nabla\Lambda}{p} \tag{54}$$

$$Y = \sigma_0\left[\frac{\Lambda}{p} - 1 - \ell^2\nabla\cdot\left(\frac{\nabla\Lambda}{p}\right)\right], \tag{55}$$

whereas the static-type higher order boundary condition reads

$$\boldsymbol{n}\cdot\boldsymbol{Y}_1 = \ell^2\boldsymbol{n}\cdot\nabla\Lambda/p = \ell^2\partial_n\Lambda/p = 0 \quad \text{on } S_f^{(1)}. \tag{56}$$

Obviously, the latter higher order boundary condition drops for a simple material ($\ell = 0$).

4.2. The plastic collapse load problem

Let the considered structure be subjected to loads that include volume forces $s\boldsymbol{b}(\boldsymbol{x})$ in V and surface tractions $s\boldsymbol{t}(\boldsymbol{x})$ on $S_f \subset S$, where $s > 0$ is a load multiplier. Guided by classical limit analysis, the set of equations which govern the plastic collapse mechanism for a material with strengthening effects can be written out as follows:

$$\nabla\cdot\boldsymbol{\sigma} + s\boldsymbol{b} = \boldsymbol{0} \text{ in } V, \quad \boldsymbol{n}\cdot\boldsymbol{\sigma} = s\boldsymbol{t} \quad \text{on } S_f \quad \text{(equilibrium cond.s)} \tag{57}$$

$$f = g(\boldsymbol{\sigma}) - Y - \sigma_0 \leqslant 0,\ \Lambda \geqslant 0,\ \Lambda f = 0 \quad \text{in } V \quad \text{(yield cond.s)} \tag{58}$$

$$\boldsymbol{E}^p = \Lambda\partial g/\partial\boldsymbol{\sigma},\ \Lambda = \|\boldsymbol{E}^p\| \quad \text{in } V \quad \text{(flow rule)} \tag{59}$$

$$\boldsymbol{E}^p = \nabla^{sym}\boldsymbol{u} \text{ in } V, \quad \boldsymbol{u} = \boldsymbol{0} \quad \text{on } S_c \quad \text{(compatibility cond.s)} \tag{60}$$

$$W(\boldsymbol{u}) := \int_V \boldsymbol{b}\cdot\boldsymbol{u}\,\mathrm{d}v + \int_{S_f}\boldsymbol{t}\cdot\boldsymbol{u}\,\mathrm{d}a = 1 \quad \text{(normalization cond.)} \tag{61}$$

$$Y = Y_0(\Lambda,\nabla\Lambda) - \nabla\cdot\boldsymbol{Y}_1(\Lambda,\nabla\Lambda) \quad \text{in } V \quad \text{(strengthening law)} \tag{62}$$

$$\Lambda = 0 \quad \text{on } S_c^{(1)}, \quad \boldsymbol{n}\cdot\boldsymbol{Y}_1 = 0 \quad \text{on } S_f^{(1)} \quad \text{(higher order bound. cond.s)}, \tag{63}$$

where Y_0 and $\boldsymbol{Y}_1$ are given by (49) and the whole volume is assumed plastically deformed in the limit state ($V_p = V$).

Here, all variables (except s) are space variables. As for their physical meaning, let us observe that the strain tensor $\boldsymbol{E}^p$ describes the shape of the collapse mechanism and is compatible with the displacement field $\boldsymbol{u}$ vanishing on $S_c := S\setminus S_f$. Since the kinematic variables (Λ, $\boldsymbol{E}^p$, $\boldsymbol{u}$) can be freely multiplied by an arbitrary positive scalar, without changing the mathematical features of the problem, and in particular without affecting the potential ψ_{st}, nor the differential form (62), follows that these variables can be usefully normalized by writing the normalization condition (61). The latter condition is well-known within classical limit analysis.

Problem equations (57)–(63) and (49) encompasses twenty-two field equations, aside an equal number of unknowns, i.e. s, $\boldsymbol{\sigma}$, $\boldsymbol{E}^p$, Λ, $\boldsymbol{u}$, Y, Y_0, $\boldsymbol{Y}_1$. This problem has a resemblance to the classical limit analysis problem in the circumstance that the elasticity law is there not involved, which means that the classical concept of rigid-plastic material is valid also in the present context. There is however a crucial difference between the above problem and the classical one. This consists in the presence of the strengthening stress, Y, in the yield function f. This stress Y is, by (62), expressed as a second order PDE in the effective plastic strain, Λ, and is accompanied by the higher order boundary conditions (63). This circumstance introduces a coupling between the static variables (s, $\boldsymbol{\sigma}$) and the kinematic ones (Λ, $\boldsymbol{E}^p$, $\boldsymbol{u}$), which does not exist in the classical limit analysis problem.

Problem equations (57)–(63) and (49) can be solved numerically with FEM procedures, like in (Anand et al., 2005) for one-dimensional shear models. In this concern one has to remark that, besides the loading data, the problem at hand possesses, as an additional piece of data, a length scale parameter, say ℓ. The latter is carried in by the strengthening stress Y, which in fact depends implicitly on ℓ, in such a way that $Y \to 0$ as $\ell \to 0$. This amounts to stating that no strengthening effects arise for a simple material ($\ell = 0$), and that therefore problem equations (57)–(62) and (49) coincides with the classical one correspondingly (note that the higher order boundary conditions drop out in the latter case). For any fixed $\ell > 0$, the above problem is expected to admit a unique solution, which should be a continuous function of ℓ. Of course, for comparison purposes particularly with experimental results, it is of high interest to investigate how this solution changes with changing ℓ, and in particular how the actual initial yield stress of the strengthening material, $\sigma_y = \sigma_0 + Y$, changes correspondingly for a given structure or specimen.

The above collapse load problem can be shown to admit a *unique solution*, which is assumed to exist. For this purpose, let the symbols $(\cdot)'$ and $(\cdot)''$ denote two different solutions that by hypothesis there exist. Then, with the notation $\Delta := (\cdot)' - (\cdot)''$, and by the virtual work principle, we can write:

$$\int_V \Delta\boldsymbol{\sigma} : \Delta\boldsymbol{E}^p\,\mathrm{d}v = \Delta s W(\Delta\boldsymbol{u}) = 0, \tag{64}$$

since $W(\Delta\boldsymbol{u}) = W(\boldsymbol{u}') - W(\boldsymbol{u}'') = 0$, that is, in extenso,

$$\int_V \boldsymbol{b}\cdot\Delta\boldsymbol{u}\,\mathrm{d}v + \int_{S_f}\boldsymbol{t}\cdot\Delta\boldsymbol{u}\,\mathrm{d}a = 0. \tag{65}$$

If $\Delta\boldsymbol{u} \not\equiv \boldsymbol{0}$, then $\boldsymbol{u}'$ and $\boldsymbol{u}''$ may differ from each other by a displacement field orthogonal to the load. We can exclude the existence of such spurious solutions and admit that $\Delta\boldsymbol{u} \equiv \boldsymbol{0}$, from where it necessarily follows that, everywhere in V, it is $\Delta\boldsymbol{E}^p \equiv \boldsymbol{0}, \Delta\Lambda \equiv 0$, $\Delta Y \equiv 0, \Delta\boldsymbol{\sigma} \equiv \boldsymbol{0}$. Therefore, the solution is unique.

5. Lower bound and upper bound theorems

In this section, the lower bound and upper bound theorems of classical plastic limit analysis are extended to the present context.

5.1. Extended lower bound theorem

We define *statically and plastically admissible load multiplier*, say $\bar{s}$, one to which we can associate a stress, a strengthening stress and some primitive strength fields, say $\bar{\boldsymbol{\sigma}}$, $\overline{Y}$, $\overline{Y}_0$, $\bar{\boldsymbol{Y}}_1$, satisfying the following conditions, that is:

$$\nabla\cdot\bar{\boldsymbol{\sigma}} + \bar{s}\boldsymbol{b} = \boldsymbol{0} \quad \text{in } V, \quad \boldsymbol{n}\cdot\bar{\boldsymbol{\sigma}} = \bar{s}\boldsymbol{t} \quad \text{on } S_f^{(1)} \tag{66}$$

$$\bar{f} := g(\bar{\boldsymbol{\sigma}}) - \overline{Y} - \sigma_0 \leqslant 0 \quad \text{in } V \tag{67}$$

$$\bar{\phi} := \phi(\overline{Y}_0, \bar{\boldsymbol{Y}}_1) \leqslant 0 \quad \text{in } V \tag{68}$$

$$\overline{Y} = \overline{Y}_0 - \nabla\cdot\bar{\boldsymbol{Y}}_1 \quad \text{in } V \tag{69}$$

$$\boldsymbol{n}\cdot\bar{\boldsymbol{Y}}_1 = 0 \quad \text{on } S_f^{(1)}. \tag{70}$$

The following *lower bound theorem* can be proved:

A statically and plastically admissible load multiplier, $\bar{s}$, cannot be larger than the collapse load multiplier, say s.

Proof. Let the state of the body in the plastic collapse be described with unmarked symbols as s, $\boldsymbol{\sigma}$, Y, $\boldsymbol{u}$, etc. Then, by Drucker's inequality we can write:

$$(\boldsymbol{\sigma}-\bar{\boldsymbol{\sigma}}):\boldsymbol{E}^{\mathrm{p}}-(Y-\overline{Y})\Lambda \geqslant 0 \quad \text{in } V. \tag{71}$$

This, by an integration over V, applying the virtual work principle and remembering (61), gives

$$s \geqslant \bar{s}+\int_V (Y-\overline{Y})\Lambda\,\mathrm{d}v. \tag{72}$$

By (69) and the analogous one for Y, we can write:

$$\begin{aligned}\int_V (Y-\overline{Y})\Lambda\,\mathrm{d}v &= \int_V [Y_0-\overline{Y}_0-\nabla\cdot(\boldsymbol{Y}_1-\bar{\boldsymbol{Y}}_1)]\Lambda\,\mathrm{d}v\\ &= \int_V [(Y_0-\overline{Y}_0)\Lambda+(\boldsymbol{Y}_1-\bar{\boldsymbol{Y}}_1)\cdot\nabla\Lambda]\,\mathrm{d}v\\ &\quad -\int_S \boldsymbol{n}\cdot(\boldsymbol{Y}_1-\bar{\boldsymbol{Y}}_1)\Lambda\,\mathrm{d}a\end{aligned} \tag{73}$$

The surface integral on the r.h. side of (73) is vanishing due to the higher order boundary conditions (70) and (63). Next, remembering (49), but with κ replaced by Λ, we can rewrite (73) as follows:

$$\begin{aligned}\int_V (Y-\overline{Y})\Lambda\,\mathrm{d}v &= \int_V \mu\left[(Y_0-\overline{Y}_0)\frac{\partial\phi}{\partial Y_0}+(\boldsymbol{Y}_1-\bar{\boldsymbol{Y}}_1)\cdot\frac{\partial\phi}{\partial \boldsymbol{Y}_1}\right]\mathrm{d}v\\ &\geqslant \int_V \mu[\phi(Y_0,\boldsymbol{Y}_1)-\phi(\overline{Y}_0,\bar{\boldsymbol{Y}}_1)]\,\mathrm{d}v.\end{aligned} \tag{74}$$

Since $\mu\phi=0,\ \mu\geqslant 0$ everywhere in V, Eq. (74) gives, by (68),

$$\int_V (Y-\overline{Y})\Lambda\,\mathrm{d}v \geqslant -\int_V \mu\phi(\overline{Y}_0,\bar{\boldsymbol{Y}}_1)\,\mathrm{d}v \geqslant 0. \tag{75}$$

Finally, as a consequence of the latter inequality, (72) implies the following one:

$$s \geqslant \bar{s}. \tag{76}$$

The set of statically and plastically admissible load multipliers include the collapse load multiplier, s, hence the equality sign in (76) is actually possible. The theorem is so proved. □

5.2. Extended upper bound theorem

Here we define *kinematically admissible load multiplier*, say $\hat{s}$, one to which we can associate some kinematic fields $\widehat{\boldsymbol{E}}^{\mathrm{p}},\ \widehat{\Lambda},\ \hat{\boldsymbol{u}}$, satisfying the equations:

$$\widehat{\boldsymbol{E}}^{\mathrm{p}}=\nabla^{\mathrm{sym}}\hat{\boldsymbol{u}} \quad \text{in } V, \quad \hat{\boldsymbol{u}}=\boldsymbol{0} \quad \text{on } S_{\mathrm{c}} \tag{77}$$

$$\widehat{\Lambda}=0 \quad \text{on } S_{\mathrm{c}}^{(1)} \tag{78}$$

$$W(\hat{\boldsymbol{u}})=1 \tag{79}$$

and has the value

$$\hat{s}:=\int_V [D(\widehat{\boldsymbol{E}}^{\mathrm{p}},\widehat{\Lambda})+\psi_{\mathrm{st}}(\widehat{\Lambda},\nabla\widehat{\Lambda})]\,\mathrm{d}v. \tag{80}$$

Hereabove, D is the dissipation function in (35), but with the plastic spin removed.

The following *upper bound theorem* can be proved:

A kinematically admissible load multiplier, $\hat{s}$, cannot be smaller than the collapse load multiplier, say s.

Proof. Let the collapse state of the structure be described by unmarked symbols, say $s,\ \boldsymbol{\sigma}$, etc. Then, by the convexity of D and ψ_{st} we can write the inequalities:

$$\left.\begin{aligned}&D(\widehat{\boldsymbol{E}}^{\mathrm{p}},\widehat{\Lambda})\geqslant D(\boldsymbol{E}^{\mathrm{p}},\Lambda)+\boldsymbol{\sigma}:(\widehat{\boldsymbol{E}}^{\mathrm{p}}-\boldsymbol{E}^{\mathrm{p}})-Y(\widehat{\Lambda}-\Lambda)\\ &\psi_{\mathrm{st}}(\widehat{\Lambda},\nabla\widehat{\Lambda})\geqslant \psi_{\mathrm{st}}(\Lambda,\nabla\Lambda)+Y_0(\widehat{\Lambda}-\Lambda)+\boldsymbol{Y}_1\cdot\nabla(\widehat{\Lambda}-\Lambda)\end{aligned}\right\}\ \text{in } V, \tag{81}$$

where the partial derivatives have been replaced with the pertaining stress and strength variables according to (36) and (49). Substituting (81) into (80) then gives

$$\begin{aligned}\hat{s}\geqslant{}&\int_V [D(\boldsymbol{E}^{\mathrm{p}},\Lambda)+\psi_{\mathrm{st}}(\Lambda,\nabla\Lambda)]\,\mathrm{d}v\\ &+\int_V \boldsymbol{\sigma}:(\widehat{\boldsymbol{E}}^{\mathrm{p}}-\boldsymbol{E}^{\mathrm{p}})\,\mathrm{d}v+\int_V(-Y+Y_0-\nabla\cdot\boldsymbol{Y}_1)(\widehat{\Lambda}-\Lambda)\,\mathrm{d}v\\ &+\int_S \boldsymbol{n}\cdot\boldsymbol{Y}_1(\widehat{\Lambda}-\Lambda)\,\mathrm{d}a.\end{aligned} \tag{82}$$

Considering that, by the virtual work principle, it is

$$\int_V \boldsymbol{\sigma}:(\widehat{\boldsymbol{E}}^{\mathrm{p}}-\boldsymbol{E}^{\mathrm{p}})\,\mathrm{d}v=s[W(\hat{\boldsymbol{u}})-W(\boldsymbol{u})]=0, \tag{83}$$

and that the last two integrals of (82) are vanishing by (62), (63) and (78), it follows that:

$$\hat{s}\geqslant \int_V [D(\boldsymbol{E}^{\mathrm{p}},\Lambda)+\psi_{\mathrm{st}}(\Lambda,\nabla\Lambda)]\,\mathrm{d}v. \tag{84}$$

Since

$$\begin{aligned}\int_V [D(\boldsymbol{E}^{\mathrm{p}},\Lambda)+\psi_{\mathrm{st}}(\Lambda,\nabla\Lambda)]\,\mathrm{d}v &= \int_V [\boldsymbol{\sigma}:\boldsymbol{E}^{\mathrm{p}}-Y\Lambda+Y_0\Lambda+\boldsymbol{Y}_1\cdot\nabla\Lambda]\,\mathrm{d}v\\ &= \int_V \boldsymbol{\sigma}:\boldsymbol{E}^{\mathrm{p}}\,\mathrm{d}v=s,\end{aligned} \tag{85}$$

it finally follows the inequality:

$$\hat{s}\geqslant s. \tag{86}$$

The theorem is so proved. □

6. Variational formulations of the plastic collapse load problem

Like in classical plastic limit analysis, the lower bound and upper bound theorems of the preceding section permit one to evaluate the collapse load multiplier, either as the maximum of lower bound values, or as the minimum of upper bound values. This gives rise to two alternative formulations.

6.1. Lower bound or static-type formulation

Basing on the theorem of Subsection 5.1, we formally can express the load multiplier, say s_{c}, as follows:

$$s_{\mathrm{c}}=\max_{(\boldsymbol{\sigma},Y,Y_0,\boldsymbol{Y}_1)} s \tag{87}$$

subject to the constraint equations (66)–(70), (without the upper bars for simplicity of writing).

We can demonstrate that the above optimization problem constitutes a true variational principle for the (plastic) collapse load problem. For this purpose, let the mentioned constraints be appended to the negative objective function, except the constraint (70), left as side condition. The augmented Lagrangian functional proves to be

$$\begin{aligned}L_1={}&-s+\int_V \boldsymbol{u}\cdot(\nabla\cdot\boldsymbol{\sigma}+s\boldsymbol{b})\,\mathrm{d}v-\int_{S_{\mathrm{f}}}\boldsymbol{u}\cdot(\boldsymbol{n}\cdot\boldsymbol{\sigma}-s\boldsymbol{t})\,\mathrm{d}a\\ &+\int_V \Lambda[g(\boldsymbol{\sigma})-Y-\sigma_0]\,\mathrm{d}v+\int_V \mu\phi(Y_0,\boldsymbol{Y}_1)\,\mathrm{d}v,\end{aligned} \tag{88}$$

where $\boldsymbol{u},\ \Lambda\geqslant 0,\ \mu\geqslant 0$ denote a displacement-like and scalar Lagrange multipliers, and the variable Y is meant to be a differential form like in (69). The first variation of (88), after some mathematics, can be cast as follows:

$$\begin{aligned}\delta L_1={}&\delta s[-1+W(\boldsymbol{u})]+\int_V \delta\mu\phi\,\mathrm{d}v+\int_V \delta Y_0\left(-\Lambda+\mu\frac{\partial\phi}{\partial Y_0}\right)\mathrm{d}v\\ &+\int_V \delta\boldsymbol{Y}_1\cdot\left(-\nabla\Lambda+\mu\frac{\partial\phi}{\partial\boldsymbol{Y}_1}\right)\mathrm{d}v+\int_S \boldsymbol{n}\cdot\delta\boldsymbol{Y}_1\Lambda\,\mathrm{d}a\\ &-\int_V \delta\boldsymbol{\sigma}:\left(\nabla^{\mathrm{sym}}\boldsymbol{u}-\Lambda\frac{\partial g}{\partial\boldsymbol{\sigma}}\right)\mathrm{d}v+\int_{S_{\mathrm{c}}}\boldsymbol{n}\cdot\delta\boldsymbol{\sigma}\cdot\boldsymbol{u}\,\mathrm{d}a.\end{aligned} \tag{89}$$

At the optimality condition, δL_1 must vanish identically for arbitrary variations, but $\delta\Lambda$ and $\delta\mu$ complying with the sign conditions $\Lambda+\delta\Lambda \geqslant 0$ and $\mu+\delta\mu \geqslant 0$ everywhere in V. This is possible if, and only if, the variables therein involved, including the Lagrange multipliers and the strain tensor $\boldsymbol{E}^{\mathrm{p}} := \Lambda\partial g/\partial\boldsymbol{\sigma}$, solve the collapse load problem. Indeed, the Lagrange multipliers take on appropriate physical meanings, $\boldsymbol{u}$ is the solution displacement field, Λ and μ the relevant consistency coefficients. The converse is also true, i.e. the solution to the collapse load problem solves the maximum problem, since in fact, in the latter case, the first variation (89) is vanishing, whereas the associated s value can be shown to be the maximum value among all statically and plastically admissible ones. The proof is complete. □

Remark 1. In the case of *no strengthening effects* (simple material), the maximum principle above coincides with its classical counterpart, as it can be easily verified.

6.2. Upper bound or kinematic-type formulation

Basing on the theorem of Subsection 5.2, the collapse load multiplier can formally be expressed as

$$s_c = \min_{(\boldsymbol{E}^{\mathrm{p}},\Lambda,\boldsymbol{u})} L_2 := \int_V [D(\boldsymbol{E}^{\mathrm{p}},\Lambda) + \psi_{\mathrm{st}}(\Lambda,\nabla\Lambda)]\,\mathrm{d}v \tag{90}$$

subject to the constraint equations (77) and (79), (but without hat signs for simplicity of writing).

The above minimization problem constitutes a true variational principle for the collapse load problem, as we prove hereafter. Proceeding as in the case of the maximum problem, let the mentioned constraints be appended to L_2. Thus we can write the augmented Lagrangian functional as

$$L_2^a = \int_V [D(\boldsymbol{E}^{\mathrm{p}},\Lambda) + \psi_{\mathrm{st}}(\Lambda,\nabla\Lambda)]\,\mathrm{d}v + s[1 - W(\boldsymbol{u})] - \int_V \boldsymbol{\sigma} : (\boldsymbol{E}^{\mathrm{p}} - \nabla^{\mathrm{sym}}\boldsymbol{u})\,\mathrm{d}v - \int_{S_c} \boldsymbol{n}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{u}\,\mathrm{d}a, \tag{91}$$

where s and $\boldsymbol{\sigma}$ are suitable scalar and stress-like Lagrange multipliers. The constraint (78) is left as side condition. No sign constraint is needed for Λ since it is incorporated in both D and ψ_{st} through their being positively homogeneous. The first variation of L_2^a reads, after some mathematics:

$$\begin{aligned}\delta L_2^a = &\int_V \delta\boldsymbol{E}^{\mathrm{p}} : \left(\frac{\partial D}{\partial \boldsymbol{E}^{\mathrm{p}}} - \boldsymbol{\sigma}\right)\mathrm{d}v + \int_V \delta\Lambda\left(\frac{\partial D}{\partial\Lambda} + \frac{\partial\psi_{\mathrm{st}}}{\partial\Lambda} - \nabla\cdot\frac{\partial\psi_{\mathrm{st}}}{\partial(\nabla\Lambda)}\right)\mathrm{d}v \\ &+ \int_S \boldsymbol{n}\cdot\frac{\partial\psi_{\mathrm{st}}}{\partial(\nabla\Lambda)}\delta\Lambda\,\mathrm{d}a - \int_V \delta\boldsymbol{u}\cdot(\nabla\cdot\boldsymbol{\sigma} + s\boldsymbol{b})\,\mathrm{d}v \\ &+ \int_{S_c} \delta\boldsymbol{u}\cdot(\boldsymbol{n}\cdot\boldsymbol{\sigma} - s\boldsymbol{t})\,\mathrm{d}a + \delta s(1 - W(\boldsymbol{u})) \\ &- \int_V \delta\boldsymbol{\sigma} : (\boldsymbol{E}^{\mathrm{p}} - \nabla^{\mathrm{sym}}\boldsymbol{u})\,\mathrm{d}v - \int_{S_c} \boldsymbol{n}\cdot\delta\boldsymbol{\sigma}\cdot\boldsymbol{u}\,\mathrm{d}a.\end{aligned} \tag{92}$$

If the set $(\boldsymbol{E}^{\mathrm{p}},\Lambda,\boldsymbol{u})$, for which δL_2^a is computed, solves the minimization problem, then δL_2^a must vanish identically for whatever variations, but $\delta\Lambda = 0$ on $S_c^{(1)}$, which leads to the Euler–Lagrange equations, i.e.

$$\boldsymbol{\sigma} = \frac{\partial D}{\partial \boldsymbol{E}^{\mathrm{p}}} \quad \text{in } V \tag{93}$$

$$Y := \frac{\partial\psi_{\mathrm{st}}}{\partial\Lambda} - \nabla\frac{\partial\psi_{\mathrm{st}}}{\partial(\nabla\Lambda)} = -\frac{\partial D}{\partial\Lambda} \quad \text{in } V \tag{94}$$

$$\nabla\cdot\boldsymbol{\sigma} + s\boldsymbol{b} = \boldsymbol{0} \quad \text{in } V, \quad \boldsymbol{n}\cdot\boldsymbol{\sigma} = s\boldsymbol{t} \quad \text{on } S_{\mathrm{f}}, \tag{95}$$

$$\boldsymbol{n}\cdot\boldsymbol{Y}_1 = 0 \quad \text{on } S_{\mathrm{f}}^{(1)} \quad \left(\boldsymbol{Y}_1 := \frac{\partial\psi_{\mathrm{st}}}{\partial(\nabla\Lambda)}\right) \tag{96}$$

besides the kinematic conditions (77) and (79). Eqs. (93) and (94) imply that $\boldsymbol{\sigma}$ and Y, being derived from the dissipation function D, satisfy the yield conditions, i.e.

$$f := g(\boldsymbol{\sigma}) - Y - \sigma_0 \leqslant 0, \quad \Lambda \geqslant 0 \quad \Lambda f = 0 \quad \text{in } V. \tag{97}$$

Also, (95) states that $\boldsymbol{\sigma}$ is a Cauchy stress in equilibrium with the load amplified by s, whereas (96) states that the higher order primitive strength, $\boldsymbol{Y}_1$, complies with the static higher order boundary condition on $S_{\mathrm{f}}^{(1)}$. Indeed, the Lagrange multipliers take on precise physical meanings. It can thus be concluded that the solution $(\boldsymbol{E}^{\mathrm{p}},\ \Lambda,\ \boldsymbol{u})$ of the minimization problem, together with the stress $\boldsymbol{\sigma}$ and the strengthening stress Y, solve the collapse load problem.

The converse is also true. In fact, denoting by $(s,\ \boldsymbol{\sigma},\ \Lambda,\ \boldsymbol{E}^{\mathrm{p}},\ \boldsymbol{u})$ the solution to the collapse load problem, we can write correspondingly:

$$L_2[\boldsymbol{E}^{\mathrm{p}} + \delta\boldsymbol{E}^{\mathrm{p}}, \Lambda + \delta\Lambda] = L_2[\boldsymbol{E}^{\mathrm{p}},\Lambda] + \delta L_2 + \delta^2 L_2, \tag{98}$$

where $\delta\boldsymbol{E}^{\mathrm{p}}$ and $\delta\Lambda$ are arbitrary variations complying with the problem's constraints, whereas δL_2 and $\delta^2 L_2$ denote the first and second variation of L_2 computed at the mentioned solution. Since $\delta L_2 = 0$ and moreover $\delta^2 L_2 > 0$ due to convexity of D and ψ_{st}, it follows:

$$L_2[\boldsymbol{E}^{\mathrm{p}} + \delta\boldsymbol{E}^{\mathrm{p}}, \Lambda + \delta\Lambda] \geqslant L_2[\boldsymbol{E}^{\mathrm{p}},\Lambda]. \tag{99}$$

This inequality holds for whatever variations $\delta\boldsymbol{E}^{\mathrm{p}}$ and $\delta\Lambda$ complying with (77) and (79), with the equality sign if, and only if, the variations are all identically null. Therefore, the considered set $(s,\ \boldsymbol{\sigma},\ \Lambda,\ \boldsymbol{E}^{\mathrm{p}},\ \boldsymbol{u})$ is also the *unique* solution to the minimization problem. The proof is so complete. □

Remark 2. If *no strengthening effects* are allowed (simple material), the minimum problem above coincides with its classical counterpart.

7. Comparison with other theories

It is useful to assess in which extent the proposed theory is able to reproduce other analogous theories of the literature.

7.1. Gradient plasticity theory by Aifantis

The gradient plasticity theory envisaged by Aifantis (1984, 1987, 2003) can be easily obtained as a particular case of the proposed one by assuming $\psi_{\mathrm{p}1} = \psi_{\mathrm{st}} = 0$ and $\psi_{\mathrm{p}2} = \frac{1}{2}\ell^2 h\|\nabla\kappa\|^2$, such that $\boldsymbol{X} = \boldsymbol{0}$, $Y = 0$, $R_0 = 0$, $\boldsymbol{R}_1 = \ell^2 h\nabla\kappa$, $R = -\ell^2 h\nabla^2\kappa$. The yield function then takes the form

$$f = g(\boldsymbol{\sigma}) - \ell^2 h\nabla^2\kappa - \sigma_0 \leqslant 0, \tag{100}$$

where ∇^2 is the Laplacian operator. The yield function (100) coincides with the one used within the theory by Aifantis.

7.2. Gradient theories based on the virtual work principle

In a recent paper (Polizzotto, 2009c) we showed that the residual-based gradient plasticity theory can be recast in a residual-free form fully equivalent to the well-known gradient plasticity theory based on the virtual work principle (VWP) (Gurtin, 2004; Gudmundson, 2004; Gurtin and Anand, 2005; Fredriksson and Gudmundson, 2005; Anand et al., 2005). It is useful to briefly resume this transformation. The key point consists in introducing the concept of *intrinsic energy production* (or *power expenditure* after Gurtin (2004)), which is here taken in the form:

$$\omega := \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{e}} + \boldsymbol{\sigma}^{\mathrm{p}} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} + \boldsymbol{\tau}^{\mathrm{p}T} \vdots \nabla\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \tag{101}$$

where $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}^{\mathrm{p}}$, $\boldsymbol{\tau}^{\mathrm{p}}$ are stresses work-conjugate of $\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}}$, $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$ and $\nabla\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$, respectively. It also consists in replacing: (i) the *insulation condition* (1) with the equivalent *energy balance condition*, that is (in the present case of free/hard boundary surface):

$$\int_V \omega \, \mathrm{d}v = \int_V \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} \, \mathrm{d}v, \tag{102}$$

and (ii) the (nonlocal) Clausius–Duhem inequality (12) with the equivalent residual-free form of it, i.e.

$$D = \omega - \dot{\psi} \geqslant 0 \quad \text{in } V, \tag{103}$$

where $\psi = \psi_e(\boldsymbol{\varepsilon}^e) + \psi_{p1}(\dot{\boldsymbol{\varepsilon}}^p, \nabla\dot{\boldsymbol{\varepsilon}}^p)$, $\psi_{p2} = \psi_{st} = 0$. Moreover, like in the VWP-based theory, the bilinear dissipation condition (2) is no longer a constitutive hypothesis (which automatically implies that $\dot{\boldsymbol{\varepsilon}}^p$ and $\nabla\dot{\boldsymbol{\varepsilon}}^p$ need being modelled independently of each other).

Since Eqs. (102) and (103) have to hold for whatever elastic/plastic deformation mechanism, we easily obtain the so-called microforces equilibrium equations and the related higher order boundary condition (Gurtin, 2004), which in the present case read:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^p - \nabla \cdot \boldsymbol{\tau}^p \quad \text{in } V \tag{104}$$

$$\dot{\boldsymbol{\varepsilon}}^p = \mathbf{0} \quad \text{on } S_c^{(1)}, \quad \boldsymbol{n} \cdot \boldsymbol{\tau}^p = 0 \quad \text{on } S_f^{(1)}, \tag{105}$$

where $\boldsymbol{\sigma} = \partial\psi_e/\partial\boldsymbol{\varepsilon}^e$, as well as the dissipation expression

$$D = \boldsymbol{\sigma}^* : \dot{\boldsymbol{\varepsilon}}^p + \boldsymbol{\tau}^{*T} \vdots \nabla\dot{\boldsymbol{\varepsilon}}^p \geqslant 0 \quad \text{in } V, \tag{106}$$

where

$$\left.\begin{aligned} &\boldsymbol{\sigma}^* = \boldsymbol{\sigma}^p - \hat{\boldsymbol{\sigma}}^p; \quad \boldsymbol{\tau}^* = \boldsymbol{\tau}^p - \hat{\boldsymbol{\tau}}^p \\ &\hat{\boldsymbol{\sigma}}^p := \partial\psi_{p1}/\partial\dot{\boldsymbol{\varepsilon}}^p, \quad \hat{\boldsymbol{\tau}}^p := \partial\psi_{p1}/\partial\nabla\dot{\boldsymbol{\varepsilon}}^p \end{aligned}\right\} \tag{107}$$

As in the VWP-based theory, the microstresses $\boldsymbol{\sigma}^p$ and $\boldsymbol{\tau}^p$ remain each without a state equation, but they can be modelled within the evolution equation formulation, for instance according to a power-law visco-plastic model as in (Gurtin, 2004; Gurtin and Anand, 2005; Anand et al., 2005; Lele and Anand, 2008). This task is achieved by writing the constitutive equations

$$\boldsymbol{\sigma}^* = \sigma_0 \left(\frac{d}{d_0}\right)^m \frac{\dot{\boldsymbol{\varepsilon}}^p}{d}, \quad \boldsymbol{\tau}^* = \ell^2 \sigma_0 \left(\frac{d}{d_0}\right)^m \frac{\nabla\dot{\boldsymbol{\varepsilon}}^p}{d}, \tag{108}$$

where d_0 is a reference effective plastic strain rate and

$$d := (\|\dot{\boldsymbol{\varepsilon}}^p\|^2 + \|\nabla\dot{\boldsymbol{\varepsilon}}^p\|^2)^{1/2}. \tag{109}$$

Substituting (108) into (104) and (107) gives

$$\boldsymbol{\sigma} - (\hat{\boldsymbol{\sigma}}^p - \nabla \cdot \hat{\boldsymbol{\tau}}^p) = \sigma_0 \left\{ \left(\frac{d}{d_0}\right)^m \frac{\dot{\boldsymbol{\varepsilon}}^p}{d} - \ell^2 \nabla \cdot \left[\left(\frac{d}{d_0}\right)^m \frac{\nabla\dot{\boldsymbol{\varepsilon}}^p}{d} \right] \right\} \tag{110}$$

$$D = \sigma_0 \left(\frac{d}{d_0}\right)^m d, \tag{111}$$

which coincide with, or are strongly similar to, analogous equations given by Anand et al. (2005), Lele and Anand (2008).

The point of main concern here is about the way the strengthening effects are extracted from (110) within the VWP-based theory. For simplicity sake, the case of one-dimensional stress/strain state is considered, such that $\dot{\boldsymbol{\varepsilon}}^p = \dot{\kappa}$. In order that the latter effects be exempt from the influence of the plastic strain rate, the power exponent m needs being taken close to zero, such as to mimic a rate-independent plasticity model. This implies that, at the limit for $m \to 0$, (110) becomes:

$$\sigma - \hat{\sigma}_{ef} = \sigma_0 \left[\frac{\dot{\kappa}}{d} - \ell^2 \nabla \cdot \left(\frac{\nabla\dot{\kappa}}{d} \right) \right], \quad \hat{\sigma}_{ef} := \hat{\sigma}^p - \nabla \cdot \hat{\tau}^p \tag{112}$$

which proves to be a *degree-zero homogeneous function* of $\dot{\kappa}$ and $\nabla\dot{\kappa}$. Furthermore, since in the plastic collapse limit state of the structure it is $\dot{\kappa} = \Lambda$, and remembering (7) and $(53)_2$, Eq. (112) is equivalent to

$$\sigma - \hat{\sigma}_{ef} = \sigma_0 \left[\frac{\Lambda}{\bar{p}} - \ell^2 \nabla \cdot \left(\frac{\nabla\Lambda}{\bar{p}} \right) \right], \quad \bar{p} = \sqrt{\Lambda^2 + \ell^2 \|\nabla\Lambda\|^2} \tag{113}$$

and thus we can set

$$Y := \sigma - \hat{\sigma}_{ef} - \sigma_0 = \sigma_0 \left[\frac{\Lambda}{\bar{p}} - 1 - \ell^2 \nabla \cdot \left(\frac{\nabla\Lambda}{\bar{p}} \right) \right], \tag{114}$$

which coincides with (56).

Our conclusion is that the capacity of the visco-plastic model, and in particular of (110), to capture strengthening effects is not due to the presence of the plastic strain rate gradient as it may appear evident at a first glance (in the limit state the plastic strain rate field changes homothetically to the plastic strain field). Instead, it depends upon the circumstance that, for $m \to 0$, Eq. (110) *tends to behave as a homogeneous function of degree zero with respect to its arguments*. This outcome is in perfect agreement with our constitutive assumptions.

8. Application

The shear model of Fig. 2 has been considered for a simple application of the nonclassical plastic limit analysis expounded previously. The problem consists in finding the collapse load $\tau_c = s_c \tau_0$, where τ_0 denotes the yield strength of the plastically undeformed material, as well as the collapse mechanism in terms of shear strain profile $\gamma_p(y)$.

The governing equations (57)–(63), noting that $\Lambda = \gamma_p \geqslant 0$ and that the yield condition is everywhere reached at collapse, can be written:

$$\frac{\gamma_p}{p} - \ell^2 \left(\frac{\gamma_p'}{p} \right)' = s_c, \quad \left(-\frac{H}{2} < y < \frac{H}{2} \right), \tag{115}$$

$$p := \sqrt{\gamma_p^2 + \ell^2 \left(\gamma_p'\right)^2}, \tag{116}$$

$$\tau_0 u(H/2) = \tau_0 \int_{-H/2}^{H/2} \gamma_p(y) \, \mathrm{d}y = \tau_0 H \quad \text{(normalization condition)}, \tag{117}$$

$$\gamma_p(-H/2) = \gamma_p(H/2) = 0, \quad u(-H/2) = 0, \tag{118}$$

where the primes denote derivative with respect to y. Eqs. (115)–(118) coincide with analogous equations given by Anand et al. (2005). The normalization condition (117), counterpart of (61), amounts to imposing a conventional unit mean shear strain, i.e. $u(H/2)/H = 1$. It is convenient to adimensionalize the above problem by introducing the new variable $\eta = y/H$, $(-1/2 \leqslant \eta \leqslant 1/2)$, and the *size ratio* $\xi := H/\ell$. Then, Eqs. (115)–(118), written for the half domain $0 \leqslant \eta \leqslant 1/2$, transform as follows:

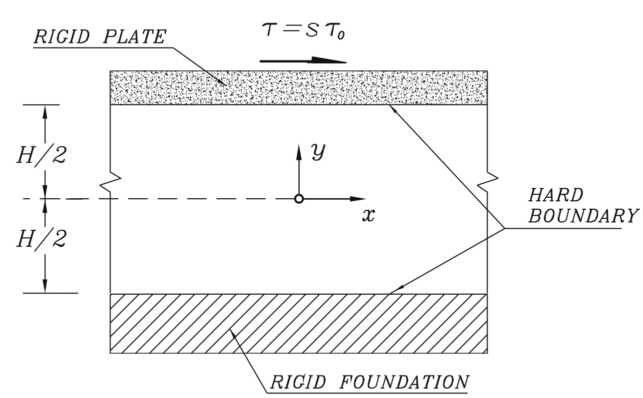

Fig. 2. Geometrical sketch of the adopted shear model.

$$\frac{\xi\gamma_p}{p}-\frac{1}{\xi}\left(\frac{\gamma_p'}{p}\right)'=s_c,\quad (0<\eta<1/2), \tag{119}$$

$$p:=\sqrt{(\xi\gamma_p)^2+(\gamma_p')^2}, \tag{120}$$

$$\int_0^{1/2}\gamma_p(\eta)\,d\eta=1/2, \tag{121}$$

$$\gamma_p'(0)=0,\quad \gamma_p(1/2)=0, \tag{122}$$

where the primes denote derivative with respect to η.

In order to solve the above differential–integral problem, the following transformation is adopted:

$$\cos\theta=\frac{\xi\gamma_p}{p},\quad \sin\theta=-\frac{\gamma_p'}{p} \tag{123}$$

such that

$$\frac{\gamma_p'}{\gamma_p}=-\xi\tan\theta, \tag{124}$$

where $\theta=\theta(\eta)$ is an unknown function. Substituting (123) into (119) gives the differential equation

$$\theta'=\xi(s_c\sec\theta-1),\quad (0<\eta<1/2) \tag{125}$$

which holds with the boundary conditions

$$\theta(0)=0,\quad \theta(1/2)=\pi/2 \tag{126}$$

corresponding, respectively, to $(122)_1$ and $(122)_2$. By integration, (125) gives

$$\theta(\eta)=\xi s_c\int_0^{\eta}\sec\theta(t)\,dt-\xi\eta, \tag{127}$$

which already satisfies $(126)_1$. On imposing condition $(126)_2$, one obtains the collapse load multiplier as

$$s_c=\frac{1+\pi/\xi}{2\int_0^{1/2}\sec\theta(t)\,dt}. \tag{128}$$

By an integration of (119) over the interval (0,1/2) one can write

$$\frac{2}{\xi}\leqslant 2\int_0^{1/2}\frac{\xi\gamma_p}{p}\,d\eta+2\left[\frac{\gamma_p'}{p}\right]_0^{1/2}\leqslant 1+\frac{2}{\xi}, \tag{129}$$

therefore s_c, as $s_c\geqslant 1$, is bounded as follows:

$$\max\left(1,\frac{2}{\xi}\right)\leqslant s_c\leqslant 1+\frac{2}{\xi}. \tag{130}$$

The upper bound in (130) coincides with a result given by Anand et al. (2005).

Next, substituting from (128) into (127) gives an integral equation as

$$\theta(\eta)=(\xi+\pi)\frac{\int_0^{\eta}\sec\theta(t)\,dt}{2\int_0^{1/2}\sec\theta(t)\,dt}-\xi\eta \tag{131}$$

which has to be used to evaluate $\theta(\eta)$. In the present application, an approximate solution to (131) has been found by an iterative procedure of the type

$$\theta_n(\eta)=(\xi+\pi)\frac{\int_0^{\eta}\sec\theta_{n-1}(t)\,dt}{2\int_0^{1/2}\sec\theta_{n-1}(t)\,dt}-\xi\eta,\quad (n=1,2,\ldots) \tag{132}$$

for a sequence of ξ values, say ξ_i, $(i=1,2,\ldots,10)$, from $\xi_1=0.5$ to $\xi_{10}=30$. The initial $(n=0)$ trial function for ξ_1 has been chosen in the form

$$\theta_0(\eta)=\pi[1-\alpha(0.5-\eta)]\eta, \tag{133}$$

with α suitably chosen within the interval (0,1), whereas for the other ξ_i, $(i=2,\ldots,10)$, it has been taken coincident with the approximate solution relative to ξ_{i-1}. With the *Mathematica* program, about 20–30 iterations were needed for every size case. Some troubles have been generated by the singularity of the integrand in (131) at the extreme $\eta=1/2$, as better explained shortly.

Next, Eq. (124) can be used to determine the shear strain γ_p. By an integration and imposing the normalization condition (121) one obtains:

$$\gamma_p(\eta)=\frac{\exp\left[-\xi\int_0^{\eta}\tan\theta(t)\,dt\right]}{2\int_0^{1/2}\exp\left[-\xi\int_0^{z}\tan\theta(t)\,dt\right]dz},\quad (0\leqslant\eta\leqslant 1/2). \tag{134}$$

The obtained results are shown in Figs. 3 and 4. Fig. 3 shows the plastic strain profiles $\gamma_p(\eta)$ for few size cases. Every profile subtends an area conventionally equal to unity, it thus represents

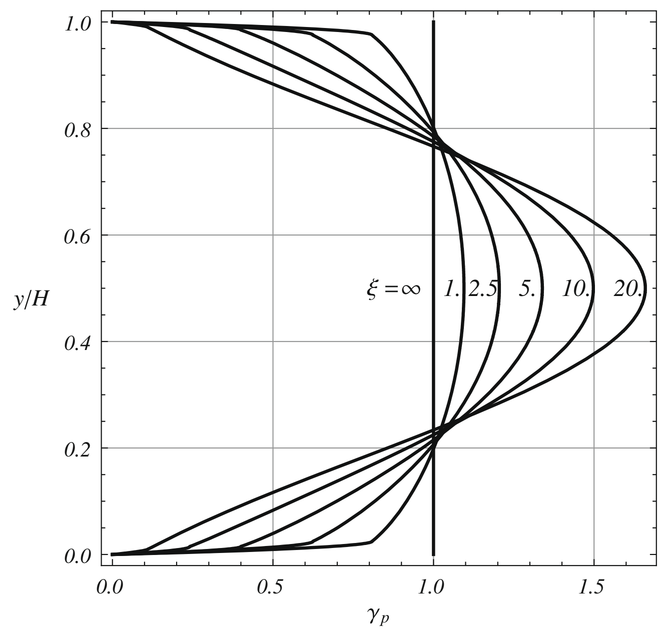

Fig. 3. Plastic shear strain profiles $\gamma_p(y/H)$ characterizing the deformation modes of the shear model at collapse for different values of the size ratio $\xi=H/\ell$.

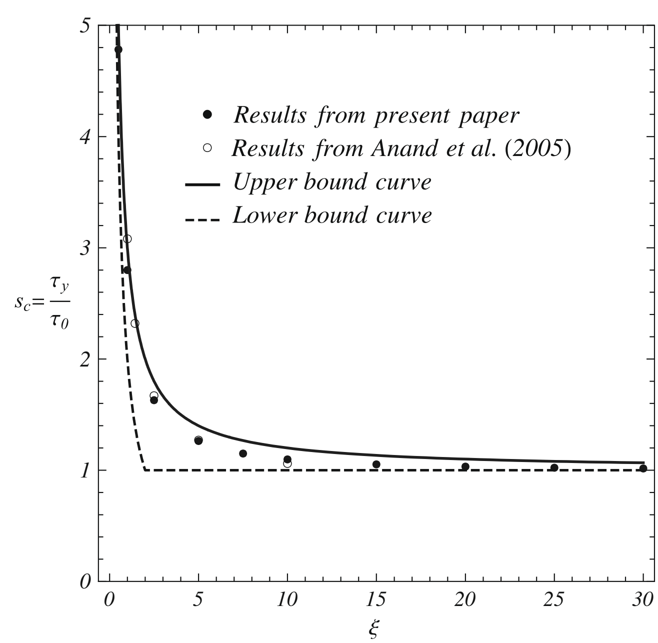

Fig. 4. Collapse load multiplier $(s_c=\tau_y/\tau_0)$/ size ratio $(\xi=H/\ell)$ diagram showing a series of the present results (filled circles) and the analogous ones by Anand et al. (2005) (empty circles), all falling between the lower and upper bound curves.

the plastic deformation mode of the relevant collapse mechanism through its own shape. The straight line profile corresponds to the classical case ($\xi = \infty$).

On comparing the latter results with the analogous ones by Anand et al. (2005), one can note some (small) differences, particularly near the boundaries. These differences are likely due to the following circumstances:

- The solutions by Anand et al. (2005) are FEM elastic–viscoplastic solutions obtained using a small value of the power-law exponent to simulate rate independency (see Fig. 3 in Anand et al. (2005)).
- The present solutions are rigid-plastic solutions, but likely affected by the disturbances caused by the singularity of the integrand of (132) mentioned above. These disturbances manifest themselves particularly close to the boundaries, where the strain profiles (exhibiting very sharp slopes) lose exactness.
- The corresponding strain profiles in the two solution series show some (small) differences in shape which are likely due to the different constitutive models therein adopted (rigid-plastic vs. elastic–viscoplastic).

Apart from these (minor) differences, the two series of results can be considered substantially in good agreement, especially if one considers that completely different solution techniques have there been used. The formation of boundary layers with sharp strain gradients, already pointed out by Anand et al. (2005), is here again detected as a basic feature of micro/nano scale structures. A wider numerical analysis for higher values of ξ would reveal how the plastic strain profiles change in shape converging (at the limit for $\xi \to \infty$)to the straight line $\gamma_p = 1$. Such analysis is here left open to a future study specifically devoted to the plastic limit analysis of one-dimensional micro/nano scale structures.

The plastic collapse load multiplier $s_c = \tau_y/\tau_0$, expressed by (128), proves to be of the form

$$s_c = a(\xi)\left(1 + \pi\frac{1}{\xi}\right), \tag{135}$$

where $a(\xi)$ is a measure of the overall plastic deformation. Since the latter parameter happens to remain almost constant on letting ξ vary within the considered interval, s_c exhibits an almost linear behavior with respect to $1/\xi = \ell/H$, hence an almost hyperbolic one with respect to $\xi = H/\ell$. The s_c values, corresponding to the considered size cases, are plotted in the s_c/ξ diagram of Fig. 4 (filled circles) together with: (i) a few analogous values given by Anand et al. (2005) (empty circles), (ii) the upper bound curve $(130)_2$ (solid line) and (iii) the lower bound curve $((130)_1$ (dashed line). The capacity of the considered shear model to strengthen with decreasing size, already pointed out in the literature (see e.g. Anand et al. (2005), Abu Al-Rub (2008) and the references therein), is demonstrated by the presented results. However, these are insufficient for a close comparison with available experimental data. This task will be addressed in a future work.

9. Conclusions

A rate-independent associative strain-gradient plasticity model has been presented, which has as a main feature its capacity to “strengthen”, i.e. to exhibit the so-called Hall–Petch effects. In this model, strengthening effects are described as the effects of a (fictitious) isotropic hardening featured by a hardening potential being a degree-one positively homogeneous (nonlinear) function of the effective plastic strain and the effective plastic strain gradient. Such a provision generates some *primitive strengths* (a scalar and a vector variables), work-conjugate of the effective plastic strain and its gradient. These two sets of variables obey a deformation-theory plasticity-like laws, specific of the strengthening material, in which a strengthening surface bounds the primitive strengths and an associative normality rule relates them to the effective plastic strain and effective plastic strain gradient. The latter law proves to be fundamental in order to obtain the *strengthening stress*, a scalar representing the actual increase (or decrease) of the initial yield strength of the material.

The proposed model differs from analogous visco-plastic models of the literature (Gurtin and Anand, 2005; Anand et al., 2005; Fredriksson and Gudmundson, 2005; Lele and Anand, 2008), in which rate-independent strengthening effects manifest themselves from a visco-plastic power law when the relevant exponent coefficient tends to the pertinent extreme value (such as to simulate rate-independent plasticity). What seems to link the two approaches with each other is that, at the limit for rate-independent plasticity, the mentioned power-law visco-plastic model generates a strengthening force which turns out to be a degree-zero positively homogeneous function of plastic strain and plastic strain gradient, much in accord with the proposed approach (at least in the cases of one-dimensional proportional plastic strain states).

The case of a perfectly plastic material with strengthening effects has been studied in some details, pointing out analogies and differences with respect to the rigid-plastic material of classical limit analysis. The following has emerged:

- As in classical limit analysis, a limit state can be envisaged in which the strengthened material structure undergoes an instantaneous plastic collapse mechanism under a constant load.
- The concept of rigid-plastic material holds true also for a strengthening material.
- The lower bound and upper bound theorems of classical plastic limit analysis can be, and in fact they have been, extended to strengthening materials, although at cost of some inevitable mathematical complications. The collapse load multiplier can still be viewed as the maximum of lower bound values, as well as the minimum of upper bound values.
- The maximum and minimum principles, direct consequences of the lower bound and upper bound theorems, respectively, represent true variational principles for the collapse load problem and are useful for numerical approximations.

As it emerges also from the simple application herein presented, the key idea of using a fictitious isotropic hardening featured by a suitable degree-one homogeneous function is a fruitful one. Other applications would be necessary to investigate various aspects of this nonclassical limit analysis (for example, what about the well-known concept of *plastic hinge*?) But, for this purpose, further study is needed in order to find out numerical methods suitable to this nonstandard (mathematically more difficult) limit analysis, with particular concern to the size dependence of the collapse load and a close comparison with the numerous available experimental data. The work by Anand et al. (2005), devoted to one-dimensional structures, provides useful hints in this regard.

An obvious extension of the results herein achieved is related to shakedown theory. Such an extension would be different from the one previously given by Polizzotto (2008b), where size dependent hardening effects were considered, but not strengthening effects, and the shakedown load was found to be size independent. In the case of strengthening effects, the shakedown load – like the plastic collapse load – is expected to be size dependent. In consideration of the importance of the provision of safety criteria for micro/nano scale structures, investigation upon this issue is of paramount interest. This will be done in a future paper.

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